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Superpotentials for M-theory on a G_2 holonomy manifold and triality symmetry

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Abstract: For M-theory on the G_2 holonomy manifold given by the cone on $\mathbf{S}^3 \times \mathbf{S}^3$ we consider the superpotential generated by membrane instantons and study its transformations properties, especially under monodromy transformations and triality symmetry. We find that the latter symmetry is, essentially, even a symmetry of the superpotential. As in Seiberg/Witten theory, where a flat bundle given by the periods of an universal elliptic curve over the u-plane occurs, here a flat bundle related to the Heisenberg group appears and the relevant universal object over the moduli space is related to hyperbolic geometry.

Keywords: M-Theory, Superstring Vacua.

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1. Introduction

The conifold transition among Calabi-Yau manifolds in type-II string theory has an asymmetrical character: an $\mathbf{S}^3$ is exchanged with an $\mathbf{S}^2$. When the situation is lifted to M-theory the resulting geometries become completely symmetrical [1, 2]: the two small resolutions given by the $\mathbf{S}^2$'s (related by a IIA flop) on the one side of the transition become $\mathbf{S}^3$'s as well (Hopf fibred by the M-theory circle $\mathbf{S}^1_{11}$; this concerns, in type-IIA language, the situation where on this side one unit of RR flux on the $\mathbf{S}^2$ is turned on and one has a $D6$ brane on the $\mathbf{S}^3$ of the deformed conifold side; cf. also [3]). This symmetry of the three $\mathbf{S}^3$'s is the triality symmetry Σ_3 of M-theory on the corresponding non-compact G_2 manifold which is a suitable deformation X_7 of a cone over $\mathbf{S}^3 \times \mathbf{S}^3 = \mathrm{SU}(2)^3/\,\mathrm{SU}(2)_D$ and comes naturally in three equivalent versions X_1, X_2, X_3.

In [2] it was pointed out that for general reasons the superpotential should have an anti-invariant behaviour under the triality symmetry, i.e. it should transform with the sign-character of Σ_3 (cf. appendix A). For this recall the action of the order two element α coming from the interchange $g_2 \leftrightarrow g_3$ of SU(2) elements in (g_1, g_2, g_3), which on $X_1 = \mathbb{R}^4 \times \mathbf{S}^3$ (when gauging g_3 to 1) is given by $(g_1, g_2) \to (g_1 g_2^{-1}, g_2^{-1})$ and induces an orientation reversing endomorphism on the tangent space (at a fixed point with $g_2 = 1$). It acts as an R-symmetry under which the superpotential transforms odd. Then a triality symmetric superpotential was considered [2] which was suggested by global symmetry considerations on the moduli space $\mathcal{N} = \mathbf{P}^1_t$ (triality symmetric behaviour meaning here that it transforms anti-invariantly, i.e. with the sign character). The simplest possibility was (with Σ_3 operating by $t \to \omega t$ where $\omega = e^{2\pi i/3}$ and $t \to 1/t$)

$$W(t) = \frac{t^3 - 1}{t^3 + 1}\,. \tag{1.1}$$

As we will show one can actually arrive at a closely related result on a different route by considering the actual non-perturbative superpotential generated by membrane instantons. The analytic continuation of these local (on $\mathcal{N}$) contributions gives an essentially symmetric superpotential (cf. below). Crucial will be a non-linear realization of the triality symmetry. Under the following action of the triality group Σ_3

$$\begin{array}{lll} z & \beta z = \dfrac{1}{1-z} & \beta^2 z = \dfrac{z-1}{z} \\ \alpha z = \dfrac{1}{z} & \alpha\beta z = 1 - z & \alpha\beta^2 z = \dfrac{z}{z-1} \end{array} \tag{1.2}$$

the holomorphic observables given by the variables η_i form a $\mathbb{Z}_3$ orbit: $\eta_{i-1} = \beta\eta_i$, $\eta_{i+1} = \beta^2\eta_i$. The variables u_i, given by the membrane instanton amplitudes and constituting local coordinates at the semiclassical ends of the global moduli space, are (when globally analytically continued; they are properly only first order variables) holomorphically related to the η_i. One has a relation $\eta_3 = \frac{1}{1-\eta_1}$ then also for the (global) u_i; note the corresponding map in [1] (t and V the sizes of $\mathbf{S}^2$ and $\mathbf{S}^3$ in type IIA; for $N = 1$ there)

$$\frac{1}{1 - e^t} \sim e^V\,. \tag{1.3}$$

We argue (in the framework of [2]) that the full multi-cover membrane instanton superpotential is given by the dilogarithm (cf. [14])

$$W(u) = \mathrm{Li}(u) = \sum_{n=1} \frac{u^n}{n^2}\,. \tag{1.4}$$

To accomplish this we reinterprete the treatment of the one-instanton amplitude in [2]. There the evaluation of the vev $\langle \int_{D_i} C \cdot \int_{Q_i} *G \rangle$ via an auxiliary classical four-form field G

was done in connection with the derivation of an ordinary interaction from the superpotential. The scalar potential computation for the superpotential can be argued to describe not only the one-instanton contribution but the full instanton series.

Remarkably, the actual superpotential given by the multi-cover membrane instantons knows 'by itself', via its analytic continuation, that it entails a triality symmetry. For the function $W(u)$ satisfies the following symmetry relations which will ensure that the dilogarithm superpotential is *compatible with triality symmetry* (i.e. that the local (on $\mathcal{N}$) membrane instanton contributions fit together globally in this sense)

$$\begin{aligned} W\left(\frac{1}{u}\right) &= -W(u) - \zeta(2) - \frac{1}{2}\log^2(-u) \\ W(1-u) &= -W(u) + \zeta(2) - \log u \log(1-u)\,. \end{aligned} \tag{1.5}$$

The symmetry relations (given here for the transformation under α and $\alpha\beta$; from these all others are derived) have the consequence that, up to the elementary corrections provided by the products of two log's and $\zeta(2)$, the $W(u)$ superpotential is invariant under the transformations in the first line of (3.3) and transforms with a minus sign under the mappings of the second line. That is the 'local' superpotential transforms (under the Sl(2) action) up to the elementary corrections with the sign character just as the global superpotential did (under the linear action) and as it should a priori. In other words triality symmetry is in this sense 'dynamical': it holds on the level of the superpotential.

Similarly, and much more trivially, the geometric world-sheet instanton series 'knows' about the $\mathbf{S}^2$-flop transition. Note the analogous behaviour of the instanton sums $I_{ws}(1/q) = -I_{ws}(q) - 1$ and (1.5) describing the multi-coverings of the supersymmetric cycles provided by the holomorphic $\mathbf{S}^2$ in the string world-sheet case and the associative $\mathbf{S}^3$ in the membrane case, respectively.

Note that when (1.4) is globally analytically continued over the critical circle (the boundary of its convergence disk) one gets monodromy contributions. The monodromy representation of the fundamental group $\pi_1(\mathbf{P}^1\backslash\{0,1,\infty\})$ will describe the multi-valuedness $\mathrm{Li}(z) \xrightarrow{l_1} \mathrm{Li}(z) - 2\pi i \log z$. The relevant local system is described by a bundle, flat with respect to a suitable connection. Like in the case of the logarithm where the monodromy of $\log z$ around $z = 0$ is captured by the monodromy matrix $M(l_0) = \begin{pmatrix} 1 & 2\pi i \\ 0 & 1 \end{pmatrix}$ and the monodromy group is given by $\mathcal{U}_{\mathbb{Z}} \hookrightarrow \mathcal{U}_{\mathbb{C}}$ where $\mathcal{U}$ denotes the upper triangular group $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \subset \mathrm{Sl}(2)$ (the embedding of $\mathcal{U}_{\mathbb{Z}}$ in $\mathcal{U}_{\mathbb{C}}$ may include here the factor of $2\pi i$), the corresponding generalisation in the case of the dilogarithm involves upper triangular 3×3 matrices, i.e. one gets again admixtures from 'lower' components when one considers constants, ordinary logarithms and the dilogarithm all at the same time. One then finds a function

$$\mathcal{L}(z) = \mathrm{Im}\,\mathrm{Li}(z) - \mathrm{Im}\log\beta z\,\mathrm{Re}\log z \tag{1.6}$$

which because of its π_1-invariance is *single-valued.* Furthermore the quantity $\mathcal{L}$ now transforms *precisely* anti-invariantly under Σ_3, i.e. without any correction terms (just as the analogous single-valued cousin $\mathrm{Re}\log z$ of the logarithm has anti-invariant transformation behaviour under the duality group $\mathbb{Z}_2$ with non-trivial element $\alpha : z \to 1/z$). So both deviations from the expected transformation properties are cured at the same time.

We will describe a number of ways to understand this anti-invariant transformation behaviour of $\mathcal{L}(z)$. Most importantly for the interpretation via a string theory duality we propose in the outlook is the geometrical interpretation as the hyperbolic volume

$$\mathrm{vol}\,\Delta(z) = \mathcal{L}(z) \tag{1.7}$$

of an ideal tetrahedron in hyperbolic three space $\mathbf{H}_3$ with vertices z_1, z_2, z_3, z_4 lying on the boundary $\mathbf{P}^1_{\mathbb{C}}$ of $\mathbf{H}_3$ which is manifestly independent of the numbering of the vertices except that the orientation changes under odd renumberings, showing the anti-invariant transformation behaviour (with z the cross ratio and $\Sigma_4 \to \Sigma_3^{\mathrm{Sl}(2)}$ after the gauging $(z_1, z_2, z_3, z_4) \to (0, 1, \infty, z)$). Corresponding to this 3-volume interpretation one has a 1-volume interpretation (in the upper half-space model for $\mathbf{H}_3$)

$$\mathrm{length}(\gamma_z) = \mathrm{Re}\log z\,. \tag{1.8}$$

Here γ_z is the path on the j-axis from j to $|z|j$. As described in the outlook it is natural to consider (1.7) and (1.8) together (cf. (6.20) and (6.21)).

It may be worth mentioning that by our description of the global relations on the quantum moduli space we get two simple reinterpretations of the membrane anomaly

$$\int_{D_1} C + \int_{D_2} C + \int_{D_3} C = \pi\,. \tag{1.9}$$

First, the non-linear $\mathrm{Sl}(2,\mathbb{Z})$ realisation (1.2) of Σ_3, which connects the three different quantities in question by a 'global' relation, makes (1.9) manifest (for z being some η_j)

$$z \cdot \beta z \cdot \beta^2 z = -1\,. \tag{1.10}$$

Moreover (1.10) is already a consequence of having a global Σ_3 symmetry at all (cf. (3.2)). And secondly, in the dual hyperbolic model a relation corresponding to (1.9) (cf. (6.7)) becomes just the angle sum in an *euclidean* triangle (the two explanations are related, cf. (C.10))

$$\alpha + \beta + \gamma = \pi\,. \tag{1.11}$$

We also compare W_{mem} to a superpotential induced by G-flux

$$W_G = \int_X (C + i\Upsilon) \wedge G\,. \tag{1.12}$$

A useful analogy is provided by the mass breaking of $N = 2$ SU(2) gauge theory by the tree-level term $W_{\mathrm{tree}} = m\,\mathrm{tr}\,\Phi^2$ whose quantum corrected version is $W_{qu} = mu$ occuring in Seiberg/Witten theory: it is given by a flux induced superpotential $W_H = \int_{\tilde{Z}} \Omega \wedge H_3$ on the Calabi-Yau $\tilde{Z}$ in type IIB (mirror dual to the type-IIA Calabi-Yau Z which describes the string embedding of the Seiberg/Witten theory) in the double scaling limit [5, 8] (crucial for this reinterpretaion is that the field theory quantity u occurs, in the appropriate limit, among the Calabi-Yau periods).

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Further we extend the theory to the case of singularities of codimension four, describing four-dimensional non-abelian gauge theories in different phases. For some further relations to type-IIA string theory and to five-dimensional Seiberg/Witten theory see [35].

For the speculative global interpretation developed in the outlook think of $\Delta(u)$, or its generalisation to a hyperbolic 3-manifold, as playing the role of the Seiberg/Witten elliptic curve E_u over the u-plane.[1] The relevant non-perturbative quantity is in both cases computed by a geometric period on the object varying over the moduli space. One might understand this as a computation of a M-theory superpotential dual to the original membrane instanton sum (analogous to the mentioned mass braking $W = mu$ in Seiberg/Witten, computed in the stringy embedding from a flux superpotential [5] where u is a period of the type-IIB mirror Calabi-Yau). If the Calabi-Yau's are $K3$ fibered over a base $\mathbf{P}^1$ then the (single) Seiberg/Witten curve can be understood as being fibered over the same $\mathbf{P}^1$ (with discrete fibre a spectral set related to $H^2(K3)$). So in the general situation of M-theory on a G_2 holonomy manifold X_7, $K3$ fibered over $\mathbf{S}^3$, one may try to compare the quantum expression given by the membrane instanton sum to a period coming from a (G-flux?) superpotential on a dual manifold Y_7 (a different $K3$ fibration over B_3, similar to some CY situations), respectively to a period in the 'thinned out' (spectral) version of Y_7 given by a hyperbolic 3-manifold M_3 (where the complexified Chern-Simons invariant is computed; this will be described in more detail elsewhere [41]).

The paper is organised as follows. In section 2 we recall, following [2], the G_2 holonomy manifold $X_7 = \mathbf{S}^3 \times \mathbb{R}^4$, the quantum moduli space $\mathcal{N}$ and the membrane anomaly. In section 3 we describe the crucial non-linear realization of the triality symmetry. In section 4 we recall the treatment of the superpotential $W(t)$ by global arguments on $\mathcal{N}$ and then describe the local approach to the superpotential by summing up the membrane instantons and investigate its deviations from strict anti-invariance with respect to the triality symmetry. Using the study of the monodromy representation (describing the Heisenberg bundle and the superpotential as its section) we describe how one can, at the cost of introducing some non-holomorphy, replace the notion of section by a function $\mathcal{L}$. $\mathcal{L}$ is shown in four ways to transform anti-invariantly; one of them uses hyperbolic geometry by giving $\mathcal{L}$ an interpretation as a hyperbolic volume. We also compare with a flux-induced superpotential. In section 5 we extend to the case of singularities of codimension four, describing four-dimensional non-abelian gauge theories in different phases. In the Outlook we compare the hyperbolic deformation moduli space with the Seiberg/Witten set-up and interprete all findings as describing a dual superpotential computation with the hyperbolic 3-simplex playing the role of the Seiberg/Witten curve. We indicate that the theory should extend to cover general $K3$ fibered *compact* G_2 manifolds (and global hyperbolic 3-manifolds). In the appendices A, B, C we study the representation of Σ_3 and give two proofs of the anti-invariance of $\mathcal{L}$. Furthermore we give some background concerning the monodromy representation of Li, the hyperbolic geometry (including the volume computation of an ideal tetrahedron), and the cohomological interpretation.

[1] Think of a different copy of H_3 over each point u in $\mathbf{P}^1$ as ambient space for $\Delta(u)$ just as one has copies of the Weierstrass embedding plane $\mathbf{P}^2_{x,y,z}$ for E_u.

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2. The G_2 manifold over $\mathbf{S}^3 \times \mathbf{S}^3$ and its moduli space

The three manifolds X_i (cf. [2]) are cones over $Y = \mathbf{S}^3 \times \mathbf{S}^3 = \mathrm{SU}(2)^3/\,\mathrm{SU}(2)_D$ where Y carries the (up to scaling) unique (Einstein) metric with $\mathrm{SU}(2)^3$ (acting from the left) and Σ_3 symmetry (da^2 stands for $-\operatorname{Tr}(a^{-1}da)^2$ where $a = g_2 g_3^{-1}, b = g_3 g_1^{-1}, c = g_1 g_2^{-1}$)

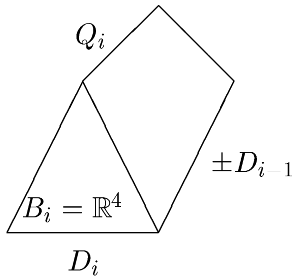
$$d\Omega^2 = \frac{1}{36}(da^2 + db^2 + dc^2)\,. \tag{2.1}$$

The images D_i in Y of the three SU(2) factors fulfill the triality symmetric relation

$$D_1 + D_2 + D_3 = 0\,. \tag{2.2}$$

(as three-cycles in homology) indicating that there are actually only two $\mathbf{S}^3$'s. The singular manifold X^{sing} is a cone over Y (r the radial coordinate)

$$ds^2 = dr^2 + r^2 d\Omega^2\,.$$



When embedded in one of the $X_i \cong \mathbb{R}^4 \times \mathbf{S}^3$ where the i'th SU(2) is filled in to a $B_i = \mathbb{R}^4$ one has $D_i \simeq 0$. The remaining three-sphere which sits at the center of X_i, corresponding to the value 0 in the $\mathbb{R}^4$ or $r = r_0$ in (2.3), is called Q_i. It is homologous to $\pm D_{i-1} \simeq \mp D_{i+1}$.

The G_2 manifold X has a covariant constant three-form Υ (resp. four-form $*\Upsilon$). The modulus $\mathrm{vol}(Q_i) \sim r_0^3$ is not dynamical but more like a coupling constant specified at infinity. The deformed manifold $X = \mathbb{R}^4 \times \mathbf{S}^3$ (which near infinity is asymptotic to (and for $r_0 \to 0$ reduces to) the cone (2.3)) has the G_2 holonomy metric ($r \in [r_0, \infty)$)

$$ds^2 = \frac{dr^2}{1-(r_0/r)^3} + \frac{r^2}{36}\left(da^2 + db^2 + dc^2 - \left(\frac{r_0}{r}\right)^3\left(da^2 - \frac{1}{2}db^2 + dc^2\right)\right). \tag{2.3}$$

Let us examine the metric perturbations which preserve G_2 holonomy. Using a new radial coordinate y with $\frac{dr^2}{1-(r_0/r)^3} = dy^2$, provided at large r, to the accuracy needed, by

$$y = r\left(1 - \frac{1}{4}\left(\frac{r_0}{r}\right)^3 + \mathcal{O}\left(\left(\frac{r_0}{r}\right)^6\right)\right), \tag{2.4}$$

one gets for the metric (with $(f_1, f_2, f_3) = (1, -2, 1)$ and up to terms $y^2\mathcal{O}((r_0/y)^6)$)

$$ds^2 = dy^2 + \frac{y^2}{36}\left(da^2 + db^2 + dc^2 - \frac{1}{2}\left(\frac{r_0}{y}\right)^3 (f_1\, da^2 + f_2\, db^2 + f_3\, dc^2)\right). \tag{2.5}$$

At small r_0 or large y one finds the conical metric with the full Σ_3 symmetry; the first correction in the expansion of powers of r_0/y (at third order) is parametrized by the f_i. So

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for (f_i) a positive multiple of $(1,-2,1)$ or its cyclic permutations *or linear combinations*[2] *of them* one gets G_2 holonomy, i.e. for (the negative f_i indicates which D_i is filled in)

$$\sum f_i = 0\,. \tag{2.6}$$

One has for the volume of Q_i and the y-dependent volume of D_i embedded in X_i (at large y; with vol D_i given up to higher order terms in $(r_0/y)^3$)

$$\begin{aligned} \text{vol } Q_i &= 2\pi^2 r_o^3 & (2.7)\\ \text{vol } D_i &= \frac{2\pi^2}{27}y^3\left(1+\frac{3}{8}f_i\left(\frac{r_o}{y}\right)^3+\mathcal{O}\left(\left(\frac{r_0}{y}\right)^6\right)\right) & (2.8)\\ &\approx \frac{2\pi^2}{27}y^3+\frac{1}{72}\,f_i\ \text{vol}\,Q_i\,. & (2.9)\end{aligned}$$

Here, the first correction to the divergent piece is the finite volume defect $\frac{1}{72}f_i\,\text{vol}(\mathbf{S}^3_{r_0})$. Note that semiclassically the volume defects are $\rho(1,-2,1)$ or a permutation (with $\rho\to\infty$; one could choose $\rho=r_0^3$ or absorb[3] ρ giving $(f_i)\sim(1,-2,1)$).

Quantum moduli space and observables. In the quantum domain there is actually [1, 2] a smooth curve $\mathcal{N}=\mathbf{P}^1_{\mathbb{C}}$ (when compactified) of theories interpolating between the three classical limits (large r_0) given by the X_i (given by three points P_i of $t=\omega^{i+1}$ with t the global coordinate, $\omega=e^{2\pi i/3}$).

A holomorphic observable on $\mathcal{N}$ must combine as SUSY partners the C-field period[4]

$$\alpha_i=\int_{D_i} C \tag{2.10}$$

with an order $1/r^3$ metric perturbation (w.r.t. the conical metric), as in

$$y_i=\exp\left(kf_i+i(\alpha_{i+1}-\alpha_{i-1})\right) \tag{2.11}$$

(with $\prod_i y_i = 1$ by (2.6)). Actually one works with the quantity

$$\eta_i=\exp\left(\frac{k}{3}(f_{i-1}-f_{i+1})+i\alpha_i\right) \tag{2.12}$$

(so $\eta_i=(y_{i-1}^2 y_i)^{1/3}, y_i=\eta_{i+1}/\eta_{i-1}$, cf. (A.11)). On the other hand one identifies η_i by global considerations on the genus zero moduli space $\mathcal{N}$ as the following rational function

$$\eta_i=-\omega\frac{t-\omega^i}{t-\omega^{i-1}} \qquad \begin{array}{|c|ccc|}\hline & P_1 & P_2 & P_3\\ \hline \eta_1 & 1 & \infty & 0\\ \eta_2 & 0 & 1 & \infty\\ \eta_3 & \infty & 0 & 1\\ \hline\end{array}\,. \tag{2.13}$$

[2]Because regarding only the lowest order term amounts to linearization of the theory; this refers thus to the situation at infinity; note that classical reasoning at infinity would expect the f_i nethertheless to be a positive multiple of a cyclic permutation of $(1,-2,1)$ to fulfill the non-linear Einstein equations in the interior; together with the other classical expectation $\int_{D_i} C=0$ (for D_i filled in) this would fix η_i to its classical value 1 at P_i; but this behaviour is modified by quantum corrections [2].

[3]In a measurement at infinity the parameter r_0 will not be known; one refers [2] also to f_i as the volume defect, when stating that the f_i go to ∞ (in ratio $(1,-2,1)$ or a permutation) for $r_0\to\infty$.

[4]Which at large radial coordinate r is independent of r for a C-field flat near infinity (to keep the energy finite), entailing that the components of C are of order $1/r^3$.

We have also the local coordinate u_i at P_i given by the membrane instanton amplitudes

$$\begin{aligned} u_i &= \exp\left(-T\mathrm{vol}(Q_i) + i\int_{Q_i} C\right) \\ \eta_i &= \exp\left(k\frac{f_i + 2f_{i-1}}{3} + i\int_{D_i} C\right). \end{aligned} \tag{2.14}$$

The local parameter u_i vanishes at P_i due to the large volume of the manifold X_i. We denote by Φ_i the physical modulus to which it is related via $u_j = e^{w_j} = e^{i\Phi_j}$, i.e. (where we have set $T = 1$; Q_i is an (isolated) supersymmetric cycle so $\Upsilon|_{Q_i}$ is the volume form)

$$\Phi_j = \int_{Q_j} C + i\Upsilon = \phi_j + i\,\mathrm{vol}(Q_j) \tag{2.15}$$

The membrane anomaly. To be well-defined the phase of the η_i variable must be modified [2] to

$$e^{i\alpha_i} = \mathrm{sign}\ \mathrm{Pf}(\mathcal{D})\ e^{i\int_{D_i} C} \tag{2.16}$$

where $\mathcal{D}$ is the Dirac operator on $\mathbf{S}^3$ with values in the positive spinor bundle of the normal bundle and Pf denotes its pfaffian (square root of the determinant) which occurs in the fermion path integral and must be combined with the classical phase factor $e^{i\int_{D_i} C}$ in the worldvolume path integral for a membrane wrapping $\mathbf{S}^3$. Now for a three-manifold X_3 which is the boundary of a (spin) four-manifold B one has [2] (with $\mathcal{D}_B$ the $S(N_B)$ valued Dirac operator on B with Atiyah-Patodi-Singer boundary conditions along X_3)

$$\mathrm{sign}\ \mathrm{Pf}(\mathcal{D}) =: e^{i\pi\mu(\mathbf{S}^3)} = e^{i\pi \mathrm{ind}(\mathcal{D}_B)/2} = e^{i\pi w_4(N_B)} = e^{i\pi\chi(B)} \tag{2.17}$$

If B could be chosen to be smooth (as for a single D_i) the correction would be ineffective, but for the union (relevant for $\sum_i \alpha_i = \pi$) of the *intersecting* D_i this cannot be the case. Now one gets the result $\sum_i \alpha_i = \pi$ either from a union of B_i's respectively bounded by the D_i or more directly from slightly perturbing the D_i as follows. For this let us recall that $\mathbf{P}^2_H = (\mathbf{H^3}\backslash\{0\})/\mathbf{H^x} = \mathbf{S}^{11}/\mathbf{S}^3$ has [2] $Y = \mathbf{S}^3 \times \mathbf{S}^3 = \mathrm{SU}(2)^3/\,\mathrm{SU}(2)_D$ fibres over a triangle $\Delta = \{[\lambda,\mu,\nu]|\lambda,\mu,\nu \geq 0\} \subset \mathbf{P}^2_{\mathbb{R}}$ from the quaternionic norm $\mathbf{P}^2_H \xrightarrow{p} \Delta = \mathbf{P}^2_{\mathbb{R}}/\mathbb{Z}_2^2$ (where $\mathbb{Z}_2^2 \cong \left((\mathbb{Z}_2^\lambda \times \mathbb{Z}_2^\mu \times \mathbb{Z}_2^\nu)/\mathbb{Z}_2^{\mathrm{diag}}\right)$. For a line $\bar{B} = \mathbf{S}^4 = \mathbf{P}^1_H \subset \mathbf{P}^2_H$ and $B = \bar{B} - \cup_{i=1}^3 D^4_{\mathrm{open}}(p_i)$ for $p_i = \bar{B} \cap L_i$ ($L_i \subset \mathbf{P}^2_H$ the coordinate lines, $D^4_{\mathrm{closed}}(p_i) \subset \bar{B}$ small 4-discs around p_i of resp. boundary $\mathbf{S}^3 \sim D_i$) one has

$$\partial B \simeq D_1 + D_2 + D_3 \tag{2.18}$$

and finds 1 for the self-intersection number (the Euler class of the normal bundle, the mod 2 relevant number). We will refer then to the membrane anomaly as

$$\sum_i \alpha_i = \pi \bmod 2\pi \qquad \text{or} \qquad \prod_i \eta_i = -1 \tag{2.19}$$

The conifold transition in type IIA. In a type-IIA reinterpretation (cf. [2]) one divides by the circle $\mathbf{S}^1_{11} = \mathcal{U}(1) \subset \mathrm{SU}(2)_1$ giving for $X_1 = \mathbb{R}^4 \times \mathbf{S}^3 = (\mathrm{SU}(2)_1 \times \mathbb{R}^{\geq 0}) \times \mathbf{S}^3$ the type-IIA manifold $(\mathbf{S}^2 \times \mathbb{R}^{\geq 0}) \times \mathbf{S}^3 = \mathbb{R}^3 \times \mathbf{S}^3$ with fixed point at the origin, i.e. the deformed conifold $T^*\mathbf{S}^3$ with a D6-brane wrapping the zero-section. For X_2 or X_3 one gets $\mathbb{R}^4 \times \mathbf{S}^3/\mathcal{U}(1) = \mathbb{R}^4 \times \mathbf{S}^2$, the two small resolutions of the conifold together with a unit of RR two-form flux on $\mathbf{S}^2$ (as $\mathbf{S}^3$ is Hopf fibered by $\mathbf{S}^1_{11}$ over $\mathbf{S}^2$). One may compare with the special lagrangian deformations [37] of the cone over T^2 with different $\mathbf{S}^1$'s killed in homology; the fixed point set under the $\mathcal{U}(1)_D$ is $L = \mathbf{S}^1 \times \mathbb{R}^2 \subset X = \mathbf{S}^3 \times \mathbb{R}^4$ (from $\mathbb{C} \subset \mathbf{H}$), the $\mathbf{S}^1$ being the boundary (where the fibre shrinks) of the disc $D^2 = \mathbf{S}^3/\mathcal{U}(1)$.

The deviation of the metric from the conical form being of order $(r_0/r)^3$ for large r (so not square-integrable in seven dimensions), r_0 is not free to fluctuate (the kinetic energy of the fluctuation would be divergent). So in the four-dimensional low energy theory it is rather a coupling constant than a modulus. To have a normalizable (or at least log normalizable) mode, one of the circles at infinity should approach a constant size (which can happen in many ways related to the Chern-Simons framing ambiguity [15])[5] which is not the case for the $\mathbb{Z}_3$ symmetric point discussed in [2] and here. In this sense the expression 'superpotential' has to be qualified; one gets the actual superpotential for an ordinary modulus if the local geometry X_7 is embedded in a compact G_2 manifold (cf. section 6.2). The metric (2.3) describes an M-theory lift of a type-IIA model with the string coupling infinite far form the $D6$ brane; to have at infinity an M-theory circle of finite radius one of the three SU(2) symmetries of (2.3) must be broken to $\mathcal{U}(1)$ [4].

3. The non-linear symmetry action

We will consider two actions of Σ_3 on $\mathbf{P}^1_{\mathbb{C}}$. In the first ('linear') action the $\mathbb{Z}_3$ sector acted by multiplication with an element of $\mathbb{C}^*$; in the second case this sector will act non-linearly (the operation of α will be given in both cases by $z \to 1/z$).

The linear action of Σ_3. Here the 'rotation' subgroup $\mathbb{Z}_3$ is generated by the action $t \to \omega t$ on $\mathcal{N} = \mathbf{P}^1_t$ and α acts by $t \to 1/t$, giving as images of t under Σ_3 $\begin{pmatrix} t & \omega t & \omega^2 t \\ t^{-1} & \omega^2 t^{-1} & \omega t^{-1} \end{pmatrix}$ (cf. (A.2)). The involution $\iota : t \to -t$ is an automorphism of $(\mathbf{P}^1_t, \Sigma_3)$, i.e.$\Sigma_3$ compatible: $\iota\gamma = \gamma\iota$.

Degenerate orbits in the t-plane. We treat the question of fixpoints or degenerate orbits. The structure of Σ_3 leads one to look for two-element and three-element orbits. The two-element orbit, whose elements are then fixed respectively by the $\mathbb{Z}_3$ cosets, is $\begin{pmatrix} 0 & 0 & 0 \\ \infty & \infty & \infty \end{pmatrix}$. In the other case the full Σ_3 orbit is already covered by the $\mathbb{Z}_3$ orbit; the transforms under the remaining (order two) elements from the non-trivial $\mathbb{Z}_3$ coset will then

[5] On $H_3(Y) = \mathbb{Z} \oplus \mathbb{Z}$ one has a full $\mathrm{Sl}(2,\mathbb{Z})$ operating but for the 'filled in' versions only the three spaces X_i are allowed as the closed and co-closed three-form Υ of class $(p,q) \in \mathbb{Z} \oplus \mathbb{Z}$ corresponds to a regular metric just for the three cases $(p,q) = (0,1)$, $(-1,0)$ or $(1,-1)$ (where the unbroken $\mathbb{Z}_2 \subset \Sigma_3$ exchanges the $\mathbf{S}^3$ factors) [16]. But for the $\mathbf{S}^1 \times \mathbf{S}^1 = T^2$ (relating to L as Y to X) not only the Σ_3 but the full $\mathrm{Sl}(2,\mathbb{Z})$ is allowed which expresses the framing ambiguity [15]. Cf. also the case $\mathbf{S}^5 \times \mathbf{S}^5$ in [17].

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just repeat the $\mathbb{Z}_3$ orbit in some order. This gives the two possibilities $\begin{pmatrix} 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}$ and $\begin{pmatrix} -1 & -\omega & -\omega^2 \\ -1 & -\omega^2 & -\omega \end{pmatrix}$. ι exchanges these two orbits and fixes the elements of the two-element orbit.

The non-linear action of Σ_3. Note that the action induced on the η_i is as follows. The cyclic permutation of the points P_i ($i = 1, 2, 3$), which is described by the rotation transformation $t \to \omega t$, produces the corresponding cyclic permutation $\eta_1 \to \eta_3 \to \eta_2 \to \eta_1$ on the η_i, as seen from (2.13). Furthermore the inversion induces $\eta_1 \to \eta_3^{-1}$, $\eta_2 \to \eta_2^{-1}$. The η_i which fulfill the relation $\eta_1\eta_2\eta_3 = -1$ (reflecting the membrane anomaly) are actually related by

$$\eta_3 = \frac{1}{1-\eta_1}\,, \qquad \eta_1 = \frac{1}{1-\eta_2}\,, \qquad \eta_2 = \frac{1}{1-\eta_3} \tag{3.1}$$

So consider now instead of the linear action of $\mathbb{Z}_3$ the non-linear action of it resp. of the full symmetry group Σ_3 as $\mathrm{Sl}(2,\mathbb{Z})/\Gamma(2)$. As is well known from the theory of the Legendre λ function, the elements are given then as the fractional linear transformations displayed below. Σ_3 occurs not only as a quotient but also as a subgroup. For this recall that the holomorphic automorphism group of $\mathbf{P}^1$ is given by $\mathrm{Aut}(\mathbf{P}^1) = \mathrm{PGl}(2,\mathbb{C}) = \mathrm{Gl}(2,\mathbb{C})/\mathbb{C}^* = \mathrm{PSl}(2,\mathbb{C}) = \mathrm{Sl}(2,\mathbb{C})/\{\pm\mathbf{1}_2\}$ and that for two triples of points of $\mathbf{P}^1$ there exists an automorphism mapping these two sets of elements onto each other. In particular we will consider the elements permuting the set $\{0, 1, \infty\}$ which are then given by transformations $z, \beta z, \beta^2 z$ and $\alpha z, \alpha\beta z, \alpha\beta^2 z$ understood as mappings $\mathbf{P}^1 \to \mathbf{P}^1$, i.e. $\mathrm{Aut}(\mathbf{P}^1, \{0, 1, \infty\}) = \Sigma_3$. This leads *just formally* to the relation (which restates (2.19))[6]

$$\prod_{i\in\mathbb{Z}_3} \beta^i z = -1\,. \tag{3.2}$$

That this product is a constant follows already from the divisor relations[7] (note that the β^i are permutations on the set $\{0, 1, \infty\}$): $(e\,z) = \underline{0} - \underline{\infty}$, $(\beta\,z) = \underline{\infty} - \underline{1}$, $(\beta^2\,z) = \underline{1} - \underline{0}$ imply that their product is a nowhere vanishing globally holomorphic function, so a constant $x \neq 0$. Now, in the $\mathbb{Z}_2$ sector given by $\{e, \alpha\}$, the transformation α, mapping $0 \leftrightarrow \infty$ and 1 to itself, will operate as multiplicative inversion (i.e. $z \cdot \alpha z = 1$). So $x^2 = \prod_{\gamma\in\Sigma_3} \gamma z = 1$ as $\prod_{i\in\mathbb{Z}_3} \beta^i z = x = \prod_{i\in\mathbb{Z}_3} \beta^i \alpha z = \prod_{i\in\mathbb{Z}_3} \alpha\beta^i z$. Then $x = (\alpha z) \cdot \beta(\alpha z) \cdot \beta^2(\alpha z) = z \cdot \alpha\beta^2 z \cdot \beta^2 z$ for an α-fixpoint (so $z = +1$ or -1) shows that $x = -1$.

One finds for the concrete functional form of the transformations

$$\begin{array}{lll} z & \beta z = \dfrac{1}{1-z} & \beta^2 z = \dfrac{z-1}{z} \\ \alpha z = \dfrac{1}{z} & \alpha\beta z = 1 - z & \alpha\beta^2 z = \dfrac{z}{z-1}\,. \end{array} \tag{3.3}$$

Degenerate orbits in the η-plane. Let us consider again the question of degenerate orbits (now under the non-linear Sl(2) action; the cases will correspond to the descriptions in the t-plane under (2.13)). Concerning first the *two-element orbits* note that their

[6]This product of the meromorphic functions $\beta^i z$ has nothing to do with the group multiplication given by composing mappings which leads to $\Sigma_3^{Sl_2}$. The point of the argument is to fix the sign-prefactors relevant for (3.2); to write down just transformations with the prescribed divisors is of course easy.

[7]So for example $(e\,z) = \underline{0} - \underline{\infty}$ indicates that the function $z \to z$ has a simple zero/pole at $0/\infty$.

elements are $\mathbb{Z}_3$ fixpoints. Therefore the condition (2.19) reads in this case $\eta^3 = -1$, so $\eta = -\omega$ or $-\omega^2$ (the solution -1 leads to another case, cf. below). So up to permutation in the still running factor $\mathbb{Z}_2 = \{e, \alpha\}$ one finds the case $\begin{pmatrix} -\omega & -\omega & -\omega \\ -\omega^2 & -\omega^2 & -\omega^2 \end{pmatrix}$. To develop the *three-element orbits* note that the three possibilities to repeat a value $\eta = e(\eta))$ from the first line in the second line (which will then, as a set, repeat the first line) lead to the cases $\eta = 1/\eta$ so $\eta = \pm 1$, or $\eta = 1 - \eta$ so $\eta = 1/2$ or ∞, or finally $\eta = \eta/(\eta - 1)$ so $\eta = 0$ or $\eta = 2$. If one looks for the corresponding orbits one finds that up to cyclic permutations (in the P_i with which one starts) one has just the two cases $\begin{pmatrix} 0 & 1 & \infty \\ \infty & 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} -1 & 1/2 & 2 \\ -1 & 2 & 1/2 \end{pmatrix}$.[8] ι exchanges these two orbits and leaves fixed the two elements of the two-element-orbit.[9]

Relations of the variables. Under the non-linear action (3.3) of the triality group Σ_3 the η_i form a $\mathbb{Z}_3$ orbit

$$\eta_{i-1} = \beta \eta_i \,, \qquad \eta_{i+1} = \beta^2 \eta_i \tag{3.4}$$

and the membrane anomaly (2.19) becomes manifest (for z some η_j)

$$\prod_{i \in \mathbb{Z}_3} \beta^i z = -1 \,. \tag{3.5}$$

As shown after (3.2) this is already a consequence of having a global Σ_3 symmetry at all.

Recall that in a semiclassical regime with $D_i = 0$ one has $Q_i \simeq \pm D_{i-1} = \mp D_{i+1}$

$$Q_i = \mp D_{i+1} \,. \tag{3.6}$$

From the classical fact (3.6) one has $\mp \int_{D_1} C = \int_{Q_3} C$, what is (with holomorphy) tantamount to saying that $\eta_{i+1} \sim u_i$ to first order (as reflected in the first order zero of η_{i+1} at P_i where u_i vanishes to first order; the respective lower sign in (3.6) is fixed at P_i). We assume that such a relation persists[10] (cf. footn.'s 11 and 13) so that one has a relation $\beta u_i = \eta_i$: the holomorphic relation between the u_i and the η_i is fixed (up to a real factor) by the relation between their arguments (imaginary parts of logarithms), so (2.14), (3.6) imply $u_i = \eta_{i+1} = \beta^2 \eta_i$, i.e.

$$\beta u_i = \eta_i \,. \tag{3.7}$$

Note that the u_i are, in contrast to the η_i, actually only first order parameters; therefore the assertion made, where we think of them as globally analytically continued, has to be suitably interpreted.[11] But note that in [14] indeed flat coordinates $-u_i$ are described (the u, v there, being shifted by πi, being the $w_i = \log u_i$ here).

[8] The ± 1 occurring here are α-fixpoints just as the elements of the two-element-orbit were $\mathbb{Z}_3$-fixpoints.

[9] $\mathbf{P}^1_\eta$ has (by transport from the $\mathbf{P}^1_t$) the involution $\iota : \eta \to \frac{\eta - 2}{2\eta - 1}$, which is $\Sigma_3^{\mathrm{Sl}(2)}$ compatible: $\iota\gamma = \gamma\iota$.

[10] Thereby a relation $\eta_{i-1} = \frac{1}{1 - \eta_i}$ holds also for the u_i: $u_3 = \beta u_1 = \frac{1}{1 - u_1}$; note the corresponding map $e^t - 1 \sim \frac{1}{e^V}$ in [1] (in the case $N = 1$ there).

[11] In the approach choosen the relation could practically be taken as a definition of $u_i^{(\mathrm{glob})}$. In principle it would be possible to rephrase the whole discussion on superpotentials to follow w.r.t. the η_i instead of $u_i^{(\mathrm{glob})}$ but it would be much less intuitive.

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(Concerning an ensuing relation (3.5) then also for the u_i note the deviation from the relation $q \cdot q^{\omega} \cdot q^{\omega^2} = +1$ which one would have in classical variables for the linear action on $q = e^s$ (cf. also footn. 18) from $s \to \omega s$ (and which would be the analogue of $q \cdot q^{-1} = 1$ in the $\mathbb{Z}_2$ case of the usual IIA flop).)[12]

We recall below that the crucial argument in [2] for a *one*-component *quantum* moduli space was the fact that the classical statement $\alpha_i = \int_{D_i} C = \mathrm{Im} \log \eta_i = 0$ is modified by membrane instantons; such a membrane instanton contribution will be present as soon as $u_i \neq 0$, i.e. away from the infinite volume limit for Q_i which would suppress the latter contribution. Similarly the classical expectation of finding the f_i in (a permutation of) the ratio $(1, -2, 1)$ and therefore having $\mathrm{Re} \log \eta_i = 0$ for one of the η_i gets modified (cf. footn. 2). So one expects that actually $u_i \neq 0 \Longrightarrow \log \eta_i \neq 0$ as is the case in (3.7).

The critical circle and special points. The 'critical' circle $|u_i| = 1$ corresponds[13] classically to the case $\mathrm{vol}(Q_i) = 0$ of a shrinking Q_i. Let us study the parameter u_i on the critical circle. One has ($\phi \in [0, 2\pi]$)

$$\log(\beta\, e^{i\phi}) = -\log\left(2 \sin\frac{\phi}{2}\right) + i\frac{(\pi - \phi)}{2}\,. \tag{3.8}$$

So here (3.7) means for $\phi_i = \int_{Q_i} C$ and[13] $\alpha_i = \int_{D_i} C$ just $\alpha_i = (\pi - \phi_i)/2$ or (cf. (2.19))

$$\int_Q C + 2\int_D C = \pi \quad (\mathrm{mod}\ 2\pi)\,. \tag{3.9}$$

Note that by (3.8) the β-transform of a parameter $z = re^{i\phi} =: e^{f+i\phi}$ on the critical circle $f = 0$ stays there in the cases (z being then a β-fixpoint by (3.9))

$$(f = 0) \qquad \phi = \pm\frac{\pi}{3}\,. \tag{3.10}$$

There are some special points in moduli space which are interesting to consider. Let us consider first the phases where the $\mathbf{S}^3$ given by Q_i has (seen classically) still physical (non-negative) volume, i.e. the domain $|u_i| \leq 1$ or $f_i \leq 0$ in parameter space[13]: here $|u_i| < 1$ (i.e.[13] $\mathrm{vol}(Q_i) > 0$) or $f_i < 0$ is the region where D_i is filled in (i.e. Q_i is not shrunken where D_i is shrinkable). So at the (classical) intersection of all three phases, where all the three-spheres have to be shrinkable at the same time, one finds $f_i = 0$, i.e. the $\mathbb{Z}_3$ orbit given by the η_i triple lies on the unit circle in the η-plane. This leads via (3.10) to the two possibilities[14]

$$f_i = 0\,, \qquad \alpha_i = \pm\frac{\pi}{3}\,. \tag{3.11}$$

[12] This s and the mentioned classical linear action is not to be confused with the t of the true global quantum moduli space and its (quantum) linear action of the same form (which corresponds under $t \leftrightarrow \eta$ to the quantum non-linear action).

[13] The geometrical interpretation applies strictly only semiclassically, in general the arguments apply just to the global variables. It is nevertheless instructive to keep this interpretation in mind.

[14] $\eta = -\omega$ and $\eta = -\omega^2$, identified above: the case where the η_i constitute a completely degenerate $\mathbb{Z}_3$ orbit ($\eta_i = \beta\eta_i = \beta^2\eta_i$), i.e. the full Σ_3 orbit degenerates to a two-element orbit.

4. The superpotential: the local approach

The global superpotential. Let us recall first the global approach to the superpotential [2]. One expects the superpotential W to vanish at the P_i. If there are no further zeroes then W will have exactly three poles on the genus zero moduli space $\mathcal{N}$. Both of these three element sets will have to be complete Σ_3 orbits *by themselves*. This leads to the two degenerate three-element orbits ω^i and $-\omega^i$. So the minimal solution is [2]

$$W \sim \frac{t^3-1}{t^3+1}\,. \tag{4.1}$$

Note that (under the linear action) it transforms with the sign character

$$W(\gamma t) = \mathrm{sign}(\gamma) W(t) \tag{4.2}$$

i.e. $W(\omega t) = W(t)$, $W(t^{-1}) = -W(t)$ (it transforms 'anti-invariantly').[15]

Concerning the zeroes and poles of the global superpotential $W(t)$ for the $\mathbb{Z}_3^{lin}$ orbits of 1 and -1 (in the t-plane), respectively, note that one has corresponding $\mathbb{Z}_3^{Sl_2}$ orbits of ± 1 (now in the η-plane!) for the η_i, and so for the u_i: $u_i = +1$ or -1, i.e.[16] $\mathrm{vol}(Q_i) = 0$ with $\int_{Q_i} C = 0$ or $\int_{Q_i} C = \pi$, so the cases are:[17] first $u_i = 0$, i.e. $\mathrm{vol}(Q_i) = \infty$ and $\int_{Q_i} C = 0$ $\Longrightarrow$ zero for W_t; and, secondly, $u_i = -1$, i.e. $\mathrm{vol}(Q_i) = 0$ and $\int_{Q_i} C = \pi \Longrightarrow$ pole for W_t (this case may be compared with the case of a vanishing $\mathbf{S}^2$ with $\int_{\mathbf{S}^2} B = \pi$).

The local superpotential. The actual superpotential arises from the sum of all the multi-cover membrane instantons

$$W(u_i) = \sum_1^\infty a_n u_i^n\,. \tag{4.3}$$

How do the non-perturbative contributions from the local semiclassical informations near the P_i fit together over the whole quantum moduli space $\mathcal{N}$? Is $W(u_i^{(\mathrm{glob})}(t)) = W(u_{i-1}^{(\mathrm{glob})}(t))$, i.e. $W(u) = W(\beta u)$: is W (at least $\mathbb{Z}_3$) triality symmetric?

First note that the membrane instantons make the deviation from the classical result $\alpha_i = 0$ possible ([2] and recalled below), just as sums of world-sheet instanton contributions in the case of the type-IIA string on a Calabi-Yau manifold give quantum corrections to a classical (complexified) Kähler volume. In the IIA case of $N = 2$ supersymmetry one can answer the analogue of our question above by considering the resummation of the geometric instanton series $I_{ws}(q) = \sum_{n\geq 1} \frac{q^n}{n^0} = \frac{q}{1-q}$ for a flop [7] where the Kähler parameter $t = \int_{\mathbf{P}^1} B + i\,\mathrm{area}(\mathbf{P}^1)$ in $q = e^{2\pi i t}$ is reflected as $t \to -t$:

$$I_{ws}\left(\frac{1}{q}\right) = -I_{ws}(q) - 1\,. \tag{4.4}$$

The deviation from anti-invariance stems from change in classical intersection numbers.

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[15] In $W(t) = \frac{t^3-1}{t^3+1}$ an underlying anti-invariant projection (A.4) is made manifest by using the relation $W(t) \sim \prod_{i\in\mathbb{Z}_3} \frac{t-\omega^i}{t+\omega^i} = \frac{1}{3}\sum_{i\in\mathbb{Z}_3} \frac{t-\omega^i}{t+\omega^i}$ giving $W(t) \sim \sum_{\gamma\in\Sigma_3} \mathrm{sign}(\gamma) \frac{1}{t+1}|_{\gamma t}$.

[16] But note here the issue of $u_i^{(\mathrm{glob})}$ vs. $u_i^{(loc)}$, cf. footn. 13 and remark after (3.7).

[17] Note that in case the independent variables z_i relevant for W build a $\mathbb{Z}_3$ orbit (like is the case for the η_i) the first case, $z_i = 0$ or $z_{i-1} = \beta z_i = 1$, differs only in the C field period, from the second one.

Now let us ask what the corresponding 'reflection' is on the modulus $\Phi_j = \int_{Q_j} C + i\Upsilon = \phi_j + i\,\mathrm{vol}(Q_j)$ in (2.15) under which we should look for reasonable transformation behaviour of the quantum corrections (reasonable meaning a way of transformation so that the three contributions fit together in a triality symmetric way over $\mathcal{N}$).

In the case of M-theory on our G_2 holonomy manifold the question of flopping an $\mathbf{S}^3$, instead of an $\mathbf{S}^2$ in type IIA on a Calabi-Yau manifold, is more complicated as the Kähler moduli no lo longer fit together naturally at the classical level [2]. Rather the metric moduli $\mathrm{vol}(\mathbf{S}^3_i)$ of the X_i, running classically over a half-line $[0, \infty)$, are at angles $2\pi/3$ to one another (in a copy of $\mathbb{R}^2$ containing the root lattice Λ of SU(3)); the C-field periods measured at infinity on the different X_i take values in *different* subgroups $E_i \cong H^3(X_i, \mathcal{U}(1)) \cong \mathcal{U}(1)$ of $H^3(Y, \mathcal{U}(1)) \cong \mathcal{U}(1) \times \mathcal{U}(1)$ (when restricted to Y).

So in view of the problems with the three rays in $\mathbb{R}^2$ we will not consider the transformation[18] given by rotation with $2\pi/3$ around the origin in this $\mathbb{R}^2$, i.e. multiplication of the modulus with ω. Rather one should now consider instead of this linear action the non-linear action (cf. (3.3)) of Σ_3 represented as $\mathrm{Sl}(2, \mathbb{Z})/\Gamma(2)$.

The one-membrane instanton contribution. Actually there is a *one-component* moduli space comprising all the P_i [2] as quantum effects given by membrane instantons cause a deviation from the classical result $\alpha_i = 0$. For this recall that to convert the interaction given by u, which is like a superpotential, to an ordinary interaction one has to integrate over the fermionic collective coordinates of the membrane instanton, i.e. to evaluate the chiral superspace integral $\int d^2\theta\, u$ (there is also an integration $\int d^4 y$ over the membrane position in $\mathbb{R}^4$ to be made). As the fermion integral has the properties of a derivation with respect to w one gets[19] ($u = e^w$; $T = 1$)

$$\int d^2\theta\, u = u \int d^2\theta\, w = -2u \int d^2\theta \int_Q \Upsilon\,. \tag{4.5}$$

For this one gets the evaluation ($w = i\Phi$, cf. (2.15))

$$\int d^2\theta\, w \sim \int_{Q_i} *G\,. \tag{4.6}$$

In a second step one finds that the contribution $\int_{\mathbb{R}^4 \times Q_i} *G$ to the effective action induces a non-zero value of $\alpha_i = \int_{D_i} C$ as one has[20]

$$\left\langle \int_{\mathbb{R}^4 \times Q_i} *_{11} G \cdot \int_{D_i} C \right\rangle \neq 0 \tag{4.7}$$

because the 'linking number' of the two three-spheres Q_i and D_i is one (effectively given by the intersection number of Q_i with B_i).

[18] Generating $\mathbb{Z}_3$, or Σ_3 when combined with complex conjugation on $\mathbb{R}^2 \cong \mathbb{C}$ (or with inversion).

[19] The contribution to $\int_D C$ of a second term $2u \int d\theta^1 w \int d\theta^2 w$ occurring here is subleading for large r.

[20] With a classical G-field generated by a source $\int_{D_i} C$ as a means to evaluate (4.7).

Derivation of the dilogarithm superpotential. The evaluation (4.7) occurred in the transition from the superpotential u to an ordinary interaction $\int d^4y\, d^2\theta\, u$: enhancing this argument we want to argue that this determines already the complete scalar potential (also suggested by the form of a G flux induced superpotential, cf. subsection 4.5), so that one gets thereby the *full* membrane instanton amplitude including all the higher wrappings, i.e. the full quantum corrections.

Note first that more generally than in (4.5) the derivative nature (w.r.t. $w = \log u$, cf. [2] p. 60) of the fermion integral gives

$$\int d^2\theta\, W(u) \approx \frac{dW}{d\log u} \int d^2\theta\, w\,. \tag{4.8}$$

Now to determine the actual sum $W(u)$ of the multi-cover membrane instantons we interpret (4.7) as representing actually a relation (with $T = 1$) between the full $\int d^2\theta\, W(u)$ (cf. footn. 48) and $\int_D C \cdot \int_Q *_7 G = \int_D C \cdot \int d^2\theta\, w$

$$\int d^2\theta\, W(u) \sim i \int_D C \cdot \int d^2\theta\, w \tag{4.9}$$

For this note that classically one has $\int_D C = 0$ and of course also $\int d^2\theta\, W = 0$ as $W = 0$. A shift $\Delta \int_D C \neq 0$ away from the classical vanishing value was argued [2] to occur via a contribution $\int d^2\theta\, W \neq 0$ from the membrane instantons. Here we argue that actually the relation between $\Delta \int_D C$ and $\Delta \int d^2\theta\, W$ should be used to show that $\int d^2\theta\, W = \frac{dW}{d\log u} \int d^2\theta\, w = \frac{dW}{d\log u} \int_Q *G = \frac{dW}{d\log u}$ *is* (proportional to) $\int_D C = \mathrm{Im}\,\log\eta$ as in (4.9). This leads with (3.7) to the differential equation[21] for W (cf. (4.60))

$$\frac{dW}{d\log u} = i \int_D C = i\ \mathrm{Im}\log\eta \overset{\text{h.c.}}{\longrightarrow}= \log\eta = \log\beta u \tag{4.10}$$

The two crucial inputs to get this were the two interpretations (3.7) and (4.9).

With the differential equation (4.10) we get for the full superpotential (cf. [13, 4])

$$W(u_i) = -\int_0^{u_i} \frac{dt}{t}\log(1-t) = \sum_{n=1}^{\infty} \frac{u_i^n}{n^2} = \mathrm{Li}_{(2)}(u_i) \tag{4.11}$$

(cf. (B.1) for the polylogarithm). Note that the integral representation (4.11) defines W on the complex plane, cut along the part $(1,\infty)$ of the positive real axis. In the series representation for large volume $\mathrm{vol}(Q_i) \approx \infty$ the instanton contributions vanish, i.e. $W(0) = 0$. The function of the modulus $\Phi = -iw = -i\log u$ has, by (4.10), a critical point exactly at $u = 0$, the large volume point P_i (cf. (4.59)). So in total[22]

$$\frac{\partial W}{\partial \Phi} = 0 \Leftrightarrow u = 0\,, \qquad W(\Phi) = 0 \Leftarrow u = 0 \tag{4.12}$$

giving no proper supersymmetric vacuum but the common decompactification runaway.

[21] Here we made a holomorphic completion on the r.h.s. which the holomorphic l.h.s. suggests.

[22] Because of the deviations (cf. below) from strict anti-invariance this captures just the $u = 0$ end.

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4.1 The triality symmetry relations of the local superpotential

Anti-invariance-with-correction-terms of the superpotential. Now, remarkably, in the case of the actual membrane instanton superpotential, the function $W(u)$ satisfies[23] the following symmetry relations which will ensure that the local superpotential is compatible with triality symmetry (almost)

$$W\left(\frac{1}{u}\right) = -W(u) - \zeta(2) - \frac{1}{2}\log^2(-u) \tag{4.13}$$
$$W(1-u) = -W(u) + \zeta(2) - \log u \log(1-u)\,. \tag{4.14}$$

The symmetry relations entail that, up to[24] the elementary corrections provided by the products of two log's and $\zeta(2)$, the $W(u)$ superpotential is invariant under the transformations in the first line of (3.3) and transforms with a minus sign under the mappings of the second line. That is the 'local' superpotential transforms (under the Sl(2) action) up to the elementary corrections with the sign character just as the global superpotential did (under the linear action) and as it should a priori. The behaviour under $\mathbb{Z}_3$ shows how the local (on $\mathcal{N}$) membrane instanton contributions fit together globally.

Relating invariance deviations by differentiation. Let us compare the analogous behaviour of the instanton sums Li_0 and Li_2, describing the multi-coverings of SUSY-cycles provided by the holomorphic $\mathbf{S}^2$ in the string world-sheet case and the associative $\mathbf{S}^3$ in the membrane case, respectively (where the 'lower terms' $-\zeta(2) - \frac{1}{2}(-\pi^2 \pm 2\pi i w)$ are at most linear in $w = \log u$)

$$W_{\mathrm{mem}}\left(\frac{1}{e^w}\right) = -W_{\mathrm{mem}}(e^w) - \frac{1}{2}w^2 + \text{lower terms} \tag{4.15}$$
$$I_{ws}\left(\frac{1}{q}\right) = -I_{ws}(q) - 1\,. \tag{4.16}$$

Note then that (4.15) corresponds after taking $\partial^2/\partial w^2$ via (B.1) to (4.16).

Let us look on a related example concerning the issue of corrections of polylogarithms. By (4.4) $\mathrm{Li}_0(q) = \sum_{n\geq 1} q^n = \frac{q}{1-q}$ had the anti-invariant transformation behaviour under $\mathbb{Z}_2$ up to mentioned correction. Said differently, when one considers the full expression which includes the classical contribution and the quantum corrections one finds a smooth behaviour.[25] The change in the classical intersection number will then be balanced exactly by the change in the quantum contribution.

[23]Integrating the relation $\frac{d}{du}W(\frac{1}{u}) = -\frac{\log(1-\frac{1}{u})}{1/u}\cdot\frac{-1}{u^2} = \frac{\log(1-u)-\log(-u)}{u}$ gives (4.13) (for the integration constant compare at $u=1$). Partial integration gives (4.14): $-\int_0^u \frac{dt}{t}\log(1-t) = -\log u \log(1-u) - \int_0^u \frac{dt}{1-t}\log t$, the last integral being $W(1-u) - W(1)$ (by the substitution $s = 1-t$ in $W(1-u)$).

[24]The differential equation (4.10) makes it technically clear that W is not precisely anti-invariant (cf. remark after (A.29)); what is remarkable is that it is almost anti-invariant.

[25]I.e. start with the prepotential as given by a cubic polynomial with the intersection numbers as coefficients (possible lower polynomial terms are not relevant here) plus the Li_3 term (including the instanton coefficients, i.e. the number of rational curves in specific cohomology classes) then take the third derivative (w.r.t. $t = \log q$) and find the classical intersection number plus $\mathrm{Li}_0(q)$ (again by (B.1)).

Now if a curve $C = \mathbf{P}^1$ is flopped at a point x_0 along the Horava/Witten intervall this is argued in [31] to cause a $G = (\pm)\delta_C$ contribution (from $dG = (\pm)\delta_C \delta(x_{11} - x_0)dx_{11}$). This comes as one has the usual anomaly balance $dG = (\operatorname{tr} F_{\text{obs/hid}} \wedge F_{\text{obs/hid}} - \frac{1}{2} \operatorname{tr} R \wedge R) \cdot \delta(x_{11} - x_{\text{obs/hid}})dx_{11}$ at the boundaries, but along the intervall, when crossing the flop point, the gravitational contribution will have changed[26] [32], with $\delta_C = \Delta_{\text{flop}} \frac{c_2}{2}$. How can one have a jump between the endpoints of the flop transition if these can also be smoothly related (when one does not go through the singular point but encircles it by rotating the B-field)? As the latter process is not just classical geometry (as would be comparing just c_2's) one has to look at the quantum corrected quantities where a classical-quantum balance now takes place at well. The relevant expression to look at is

$$12\, F_1 = \left(\frac{c_2}{2} \cdot J \right) t + \mathrm{Li}_1(q) \tag{4.17}$$

and $\partial_t F_1 \sim \frac{c_2}{2} \cdot J + \mathrm{Li}_0(q)$ brings us effectively back to the previous balancing argument.

Anti-invariance of the superpotential-with-correction-terms? One might ask whether one could make the superpotential anti-invariant by adding suitable terms to W so that the troublesome remainder terms[27] $R_\gamma(z)$ (for $\gamma \in \Sigma_3$) in

$$W(\gamma\, z) = \mathrm{sign}(\gamma) W(z) + R_\gamma(z) \tag{4.18}$$

(the log's and constants interfering with the precise anti-invariant transformation behaviour) would be canceled. For example the deviation in (4.16) can be rectified that way, the modified $\tilde{I}_{ws} := I_{ws} + 1/2$ is strictly anti-invariant: $\tilde{I}_{ws}(1/q) = -\tilde{I}_{ws}(q)$. As for W the deviations consist of quadratic polynomials in the log's of u and its $\mathbb{Z}_3^{\mathrm{Sl}_2}$ transforms one may try to adjust W by adding a term of this type (as suggested by the fact that I_{ws} and W_{mem} are related by taking a two-fold derivative w.r.t. $\log u$, cf. above).

So one would make an ansatz to correct the superpotential by additional terms[28] $C(z)$ which cancel the unwanted terms. Including $C(z)$ one gets a modified superpotential

$$\tilde{W}(z) = W(z) + C(z) \tag{4.19}$$

such that $\tilde{W}$ transforms just with the sign character. One expects $\tilde{W}$ to have the structure making manifest an underlying anti-invariant projection (A.4) (cf. appendix A.3)

$$\frac{1}{6} \sum_{\gamma \in \Sigma_3} \mathrm{sign}(\gamma) W(\gamma \cdot) = \frac{1}{6} \sum_{\gamma \in \Sigma_3} \mathrm{sign}(\gamma) [\mathrm{sign}(\gamma) W + R_\gamma] = W + \frac{1}{6} \sum_{\gamma \in \Sigma_3} \mathrm{sign}(\gamma) R_\gamma\, . \tag{4.20}$$

[26] The reason for the 1/2 is that $c_2(CY) \cdot S = c_2(S) - c_1^2(S)$ changes by 2 as an blow-up increases the Euler number of S by one and the canonical class gets a contribution from the exceptional divisor.

[27] From R_α and $R_{\alpha\beta}$ in (4.13) and (4.14) one derives iteratively that $R_\beta = -2\zeta(2) - \log z \log \beta z - \frac{1}{2} \log^2(-\beta z)$, $R_{\beta^2} = -\zeta(2) - \log z \log \beta z - \frac{1}{2} \log^2 z$, $R_{\alpha\beta^2} = -\frac{1}{2} \log^2(1-z)$; $R_{\alpha\beta^2}$ follows also directly from integrating $\frac{d}{dz} W(\frac{z}{z-1}) = -\frac{\log(1-\frac{z}{z-1})}{\frac{z}{z-1}} \frac{-1}{(z-1)^2} = \log(1-z) \left(\frac{1}{z} + \frac{1}{1-z} \right)$.

[28] Reasonably one has to demand that $C(z)$ is 'more elementary' than $W(z)$ itself (like a polynomial in log terms) as one has always the anti-invariant projection (A.4) with $C = \sum_{\gamma \in \Sigma_3, \gamma \neq e} \mathrm{sign}(\gamma) \gamma W$. From $C \sim \sum \mathrm{sign}(\gamma) R_\gamma$ as used above one finds that for this it suffices that the R_γ are 'more elementary'.

From this one has for the sought-after correction term[29] $C = \frac{1}{6}\sum_{\gamma\in\Sigma_3}\mathrm{sign}(\gamma)R_\gamma$

$$C = \frac{1}{6}\Big(-\frac{1}{2}\log^2 z - 2\log z\log\beta z + \frac{3}{2}\log^2\beta z + 2\log\beta z\log\beta^2 z + \frac{1}{2}\log^2\beta^2 z\Big)\,. \qquad (4.21)$$

One finds also (with a subtle sign and remote similarity to the Rogers modification (B.16))

$$\tilde{W}(z) = \mathrm{Li}(z) - \frac{1}{2}\log\beta z\log z - \frac{\pi^2}{6} \pm \frac{i\pi}{6}(\log z - \log\beta z)\,. \qquad (4.22)$$

But a physical motivation for C is unclear; one wants to see the local individual superpotentials in the three different phases not added like in (4.20) but rather naturally patching together (cf. remark after (4.3)).

Behaviour of the local superpotential on the critical circle. Let us study W on[30] the critical circle $|u_i| = 1$ (the boundary of the domain of convergence of the series (4.11)), so just as a function of $\phi_i = \int_{Q_i} C$. Then $W(e^{i\phi}) - W(1) = -i\int_0^\phi d\chi\log(1-e^{i\chi})$ and (3.8) give the elementary evaluation

$$\mathrm{Re}\,W(e^{i\phi}) = \sum_{n\geq 1}\frac{\cos n\phi}{n^2} = \zeta(2) - \frac{1}{4}\phi(2\pi-\phi) \qquad (4.23)$$

for the real part and the non-elementary odd function $I(\phi)$ for the imaginary part

$$I(\phi) = \sum_{n\geq 1}\frac{\sin n\phi}{n^2} = -\int_0^\phi \log\left(2\sin\frac{\psi}{2}\right)d\psi \qquad (4.24)$$

which is of period[31] 2π. As we will have opportunity to consider terms like

$$\Pi(\phi) := \frac{1}{2}I(2\phi) = -\int_0^\phi \log(2\sin\psi)d\psi \qquad (4.25)$$

let us note that[32]

$$\frac{1}{2}I(2\phi) = I(\phi) + I(\pi+\phi) = I(\phi) - I(\pi-\phi) \qquad (4.26)$$

[29] Coming with (A.15) from $6C = -2\log z\log\beta z + \log z\log\beta^2 z + \log^2(-z) + \frac{1}{2}\log^2\beta z - \frac{1}{2}\log^2(-\beta z)$.

[30] Which would correspond classically[13] to a shrinking Q_i of $\mathrm{vol}(Q_i) = 0$.

[31] If $\phi\notin[0,2\pi]$ one has to take the absolute value inside the logarithm.

[32] As

$$\begin{aligned} I(2\phi) &= -2\int_0^\phi d\chi\log(2\sin\chi) \\ &= -2\int_0^\phi d\chi\log\left(2\sin\frac{\chi}{2}\right) - 2\int_0^\phi d\chi\log\left(2\cos\frac{\chi}{2}\right) \\ &= 2I(\phi) + 2\int_\pi^{\pi-\phi} d\xi\log\left(2\sin\frac{\xi}{2}\right) \\ &= 2I(\phi) + 2I(\pi) - 2I(\pi-\phi)\,. \end{aligned}$$

In view of the membrane anomaly relation $\alpha_1 + \alpha_2 + \alpha_3 = \pi$ (for the D_i) with symmetry for $\alpha_i = \pi/3 = \arg(-\omega^2)$ or $\alpha_i = -\pi/3 = \arg(-\omega)$, note[33] that $I(\phi)$ becomes maximal at $\phi = \pi/3$ (as seen from solving $I'(\phi) = -\log(2\sin\frac{\phi}{2}) = 0$; cf. (3.11)) and that (by (4.26))

$$I(\pi/3) = \frac{3}{2} I(2\pi/3) \tag{4.27}$$

So for the six roots $e^{k\frac{2\pi i}{6}}$ $(k = 1, \ldots, 6)$ one has $(I := I(\pi/3),\ \zeta := \zeta(2) = \pi^2/6)$

$e^{k\frac{2\pi i}{6}}$	$-\omega^2$	ω	-1	ω^2	$-\omega$	1
$W(e^{k\frac{2\pi i}{6}})$	$\frac{1}{6}\zeta + iI$	$-\frac{2}{6}\zeta + \frac{2}{3}iI$	$-\frac{3}{6}\zeta$	$-\frac{2}{6}\zeta - \frac{2}{3}iI$	$\frac{1}{6}\zeta - iI$	ζ

(4.28)

A $\mathbb{Z}_N$ symmetry property. The result that $\sum_{\mathbb{Z}_6} \mathrm{Li}(e^{k2\pi i/6}) = \zeta/6 = \mathrm{Li}(1)/6$ leads to a more general observation concerning the angular degree of freedom ϕ of $\mathrm{Li}(z)$, more precisely on the interrelation between entries equidistributed with respect to ϕ. Generally one has[34] (with $\omega_N = e^{2\pi i/N}$)

$$\frac{1}{N}\mathrm{Li}(z^N) = \sum_{k\in\mathbb{Z}_N} \mathrm{Li}(\omega_n^k z)\,. \tag{4.29}$$

4.2 The monodromy representation

The multi-valuedness of $\log z$ and $\mathrm{Li}(z)$ around $z = 0, 1$ and ∞ is described by the monodromy representation of the fundamental group $\pi_1(\mathbf{P}^1\backslash\{0,1,\infty\})$. This describes for the generator loops $l_i(t)$ $(i = 0, 1,\ t \in [0,1])$ which encircle (in the mathematically positively oriented sense) $z = 0$ and $z = 1$, respectively (then $l_\infty \circ l_1 \circ l_0 = 1$), the increments (cf. also appendix B)

$$\log z \xrightarrow{l_0} \log z + 2\pi i \tag{4.30}$$

$$\log \beta z \xrightarrow{l_1} \log \beta z - 2\pi i\,, \qquad \mathrm{Li}(z) \xrightarrow{l_1} \mathrm{Li}(z) - 2\pi i \log z\,. \tag{4.31}$$

The relevant local system is described by a bundle, flat with respect to a suitable connection. In the case of the logarithm the monodromy (4.30) is captured by the matrix $M(l_0) = \begin{pmatrix} 1 & 2\pi i \\ 0 & 1 \end{pmatrix}$ acting on the two-vector $(\log z, 1)^t$ and the monodromy group is given by $\mathcal{U}_\mathbb{Z} \hookrightarrow \mathcal{U}_\mathbb{C}$ where $\mathcal{U}$ denotes the upper triangular group $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \subset \mathrm{Sl}(2)$ (the embedding of $\mathcal{U}_\mathbb{Z}$

[33] For the Q_i but in view of the identification[11, 13] (3.7) of the D_i and the Q_i in a $\mathbb{Z}_3$ rotated phase.

[34] As $1 - y^N = \prod_k(\omega_N^k - y) = \prod_k(1 - \omega_N^{-k} y)$ gives

$$\begin{aligned} -\frac{1}{N}\int_0^{z^N} \log(1-t)d\log t &= -\int_0^z \log(1-y^N)d\log y \\ &= -\sum_k \int_0^z \log(1-\omega_N^{-k}y)d\log y = -\sum_k \int_0^z \log(1-\omega_N^k x)d\log x \\ &= -\sum_k \int_0^{\omega_N^k z} \log(1-t)d\log t\,. \end{aligned}$$

in $\mathcal{U}_{\mathbb{C}}$ may include the factor of $2\pi i$). The generalisation in the case of the dilogarithm involves the upper triangular 3×3 matrices [18], i.e. one gets again admixtures from 'lower' components: the hierarchical structure of the poly-logarithm $\mathrm{Li} = \mathrm{Li}_2$ with respect to its predecessor $\log\beta z = \mathrm{Li}_1(z)$ entails that its monodromy is not any longer given just by the addition of integers (multiplied by $2\pi i$); rather one has to consider constants, ordinary logarithms and the dilogarithm all at the same time and to consider the lower ones as monodromy contributions of the next higher one. One can organize this as follows. Analytic continuation about a loop l in $\mathbf{P}^1\backslash\{0,1,\infty\}$ (based at $1/2$, say) leads to the monodromy representation

$$M:\pi_1(\mathbf{P}^1\backslash\{0,1,\infty\})\to \mathrm{Gl}(3,\mathbb{C})\,. \tag{4.32}$$

There are two equivalent ways to express this. In a *vector picture* one assembles Li, the ordinary logarithm and the constants to a three-vector c_3 and finds for the images of the generator loops $l_i(t)$ $(i=0,1)$ the matrices $M(l_i)$ representing (4.30), (4.31) for c_3

$$c_3=\begin{pmatrix}\mathrm{Li}(z)\\ \log z\\ 1\end{pmatrix}:\qquad M(l_0)=\begin{pmatrix}1&0&0\\0&1&2\pi i\\0&0&1\end{pmatrix},\qquad M(l_1)=\begin{pmatrix}1&-2\pi i&0\\0&1&0\\0&0&1\end{pmatrix}. \tag{4.33}$$

Alternatively, in a *Heisenberg picture*, consider the complexified Heisenberg group $\mathcal{H}_{\mathbf{C}}$

$$\begin{pmatrix}1&a&c\\0&1&b\\0&0&1\end{pmatrix}\xrightarrow{\cong}(a,b\,|\,c)\in\mathcal{H}_{\mathbf{C}}\,. \tag{4.34}$$

Instead of c_3 one considers here the expression (a flat section of a suitable connection)

$$\Lambda(z)=\Big(-\log\beta z,\log z\,|\,-\mathrm{Li}(z)\Big) \tag{4.35}$$

and left operation with $\mathcal{H}_{\mathbf{Z}}$ expresses the multi-valuedness (4.31). More precisely one finds for the monodromy along the loops l_i the representing left multipliers h_i

$$h_0=(0,1\,|\,0)\,,\qquad h_1=(1,0\,|\,0)\,,\qquad h_\infty=(-1,-1\,|\,0)\,. \tag{4.36}$$

So $h_0\cdot(u,v\,|\,w)=(u,v+1\,|\,w)$ and $h_1\cdot(u,v\,|\,w)=(u+1,v\,|\,w+v)$ give[35] (4.31).

One has from $(a,b\,|\,c)\to(x,y)=(e^a,e^b)$ a bundle projection with fibre $(2\pi i)^2\mathbb{Z}\backslash\mathbb{C}_c\xrightarrow{\cong}\mathbb{C}^*$ (via $c\to S:=e^{c/2\pi i}$; the entries of $\mathcal{H}_{\mathbb{Z}}$ are actually from $((2\pi i)\mathbb{Z},(2\pi i)\mathbb{Z}\,|\,(2\pi i)^2\mathbb{Z})$)

$$\begin{array}{c}\mathcal{H}_{\mathbb{Z}}\backslash\mathcal{H}_{\mathbb{C}}\\ \downarrow\\ \mathbb{C}^*_x\times\mathbb{C}^*_y=(2\pi i\mathbb{Z})^2\backslash\mathbb{C}^2_{a,b}\,.\end{array} \tag{4.37}$$

[35]Note that actually we consider $(a,b,c)\in\mathbb{Z}^3$ embedded in $\mathcal{H}_{\mathbf{Z}}$ via $(2\pi ia,2\pi ib\,|\,(2\pi i)^2c)$.

This carries a connection of curvature $\frac{1}{2\pi i} d\log x \wedge d\log y$ coming from the connection

$$\nabla S = \frac{1}{2\pi i} S(2\pi i d\,\log S - udv) = dS - S\,u\,dv/2\pi i \tag{4.38}$$

on the pullback[36] of the bundle (4.37) to $\mathbb{C}\times\mathbb{C}$ along $(a,b) \to (e^a, e^b)$. The latter trivialises the bundle so that a section can be understood as a map $S : \mathbb{C}\times\mathbb{C} \to \mathbb{C}^*$.

Now consider the pullback (along the base map $z \to (1-z, z)$) of the bundle $\mathcal{H}_\mathbb{Z}\backslash\mathcal{H}_\mathbb{C}$ lying over $\mathbb{C}^*\times\mathbb{C}^*$ to what we will call the *Heisenberg bundle* $\underline{\mathcal{H}}$ over $\mathbf{P}^1\backslash\{0,1,\infty\}$

$$\begin{array}{ccc} \underline{\mathcal{H}} & \longrightarrow & \mathcal{H}_\mathbb{Z}\backslash\mathcal{H}_\mathbf{C} \\ \downarrow & & \downarrow \\ \mathbf{P}^1\backslash\{0,1,\infty\} & \stackrel{(1-z,z)}{\longrightarrow} & \mathbb{C}^*\times\mathbb{C}^*\,. \end{array} \tag{4.39}$$

As the first two entries of a section s of $\underline{\mathcal{H}}$ are fixed by the construction (up to the indeterminacy caused by the coset) s has the form $s(z) = \mathcal{H}_\mathbb{Z}(-\log\beta z, \log z\,|\,c)$. Asking even for a *flat* section one finds (undoing the fibre identification $c \to e^{c/2\pi i} = S$) that the flatness condition $dc = u\,dv$ (from (4.38)) just expresses (4.10), the Li integral, and that the coset takes into account the multi-valuedness (4.31). So $\underline{\mathcal{H}}$ possesses the *flat* section (4.35) and the Heisenberg bundle just encodes the fact that the 'function' Li is a section.

To gain information about Li itself by somehow projecting to it is not straightforward as the immediate extraction of Li is obstructed by the $\mathcal{H}_\mathbb{Z}$ coset. What actually can be extracted is the (suitably adjusted) imaginary part of it as we describe now.

4.3 Anti-invariance of the adjusted imaginary part $\mathcal{L}$ of W

To motivate this note that the integral representation (4.11) defines W on the complex plane, cut along the part $(1,\infty)$ of the positive real axis where W jumps by $2\pi i\log z$ when crossing the cut; so the expression $W(z) + i\arg(1-z)\log z$ is continuous; its imaginary part coincides with $\mathcal{L}$ below.

Now note that the *complex-valued* 'function' $\mathrm{Li}(z)$ of the *complex* variable z is not well-defined as a function according to the multi-valuedness expressed by the increments Δ_i around the l_i which follow from (4.31), i.e. $\Delta_0 = 0$, $\Delta_1 = -2\pi i\log z$. Note that if we restrict the values by considering just $\mathrm{Im}\,\mathrm{Li}(z)$ then this *real-valued* 'function' of the *complex* variable z has still $\Delta_0 = 0$, $\Delta_1 = -2\pi\,\mathrm{Re}\log z$. Therefore, if we go one step further and consider the *real-valued* 'function' of the *real* degree of freedom $z = e^{i\phi}$ living on the critical circle $|z| = 1$, we get indeed a well-defined function.

Now it is interesting to see that, with a slight modification, we can actually do better. Namely, to extrapolate this property beyond the critical circle, consider the expression $\psi = \log\beta z\;\mathrm{Re}\log z$ (vanishing on $|z|=1$). One finds that the real-valued combination of a complex degree of freedom $\mathcal{L}(z) = \mathrm{Im}\;\mathrm{Li}(z) - \mathrm{Im}\;\psi(z)$ is actually not only a well-defined function i.e. $\pi_1(\mathbf{P}^1\backslash\{0,1,\infty\})$-invariant, but at the same time also Σ_3 anti-invariant, so it transforms *precisely* with the sign-character, i.e. without correction terms. Furthermore, it is even expressible by a function depending just on a *real* degree of freedom: the critical circle.

[36] The bundle $(2\pi i)^2\mathbb{Z}\backslash\mathcal{H} \to \mathbb{C}_a \times \mathbb{C}_b$ of fibre $(2\pi i)^2\mathbb{Z}\backslash\mathbb{C}_c$; so this is the complex analogue of (B.7).

This is the case although $\mathcal{L}$ is not just depending only on the angular part of the complex variable z; rather it depends on the value of $I(\phi) = \mathrm{Im\ Li}|_{e^{i\phi}} = \mathcal{L}|_{e^{i\phi}}$ on the angular parts of the different $\mathbb{Z}_3$ transforms of z, which themselves are not depending on the angular part of z alone, cf. (4.42).

So note first that the function one finds (which also satisfies $\mathcal{L}(\bar{z}) = -\mathcal{L}(z)$)

$$\mathcal{L}(z) = \mathrm{Im\,Li}(z) - \mathrm{Im}\log\beta z\,\mathrm{Re}\log z \tag{4.40}$$

(cf. (B.14)) is π_1-invariant, i.e. *single-valued* as is also easily checked from (4.31).

Now, just as the single-valued cousin $\mathrm{Re}\log z$ of the logarithm has anti-invariant transformation behavior under the duality group $\mathbb{Z}_2$ (with $\alpha : z \to 1/z$), we will see that $\mathcal{L}$ transfroms anti-invariantly under Σ_3. We give four arguments for this: the direct computational check, an argument using representation theory, a manifestly invariant rewriting and finally a geometric interpretation.

Anti-invariance of $\mathcal{L}$ (first argument): explicit evaluation (appendix A.1). This is the brute force procedure given by the explicit check.

Anti-invariance of $\mathcal{L}$ (second argument): representation theory (appendix A.2). More conceptually one has a representation theoretic argument (cf. (A.24), (A.27)).

Anti-invariance of $\mathcal{L}$ (third argument): rewriting to a manifestly invariant expression. The third argument (going in essence back to Kummer) is by an explicit rewriting (4.43). Consider the decomposition of $\mathrm{Li}(z)$ in real and imaginary parts as we did above for its restriction on the critical circle $|z| = 1$. One finds with $z = re^{i\phi}$ that

$$\begin{aligned}\mathrm{Li}(z) &= -\int_0^r \frac{\log(1-e^{i\phi}t)}{t}dt \\ &= -\frac{1}{2}\int_0^r \frac{\log(1-2t\cos\phi+t^2)}{t}dt + i\int_0^r \arctan\left(\frac{t\sin\phi}{1-t\cos\phi}\right)\frac{dt}{t}\,.\end{aligned} \tag{4.41}$$

This gives[37] with $\arctan\frac{t\sin\phi}{1-t\cos\phi} =: \chi$, $\kappa := \chi|_{t=r} = \mathrm{Im}\log\beta z$ and the inversion relation $t = \frac{\sin\chi}{\sin(\chi+\phi)}$ (considering ϕ as a parameter) the evaluation (using (4.25), (A.13))

$$\begin{aligned}\mathrm{Im\,Li}(z) &= \int_0^r \chi\frac{dt}{t} = \chi\log t|_0^r - \int_0^\kappa \log t\,d\chi = \kappa\log r - \int_0^\kappa \log\frac{\sin\chi}{\sin(\chi+\phi)}d\chi \\ &= \kappa\log r + \frac{1}{2}\Big(I(2\phi) + I(2\kappa) - I(2\phi+2\kappa)\Big) \\ &= \mathrm{Im}\log\beta z\,\mathrm{Re}\log z + \frac{1}{2}\Big(I(2\,\mathrm{Im}\log z) + I(2\,\mathrm{Im}\log\beta z) + I(2\,\mathrm{Im}\log\beta^2 z)\Big)\,.\end{aligned} \tag{4.42}$$

This shows that the 'function' Im Li(z), which a priori is a non-elementary real 'function' of a *complex* variable, is actually already determined (up to the elementary logarithmic product term) by the real 'function' $I(\phi)$ of a *real* degree of freedom (cf. remark above).

[37]Note that $\mathrm{Im}\log z = \arctan\frac{\mathrm{Im}\,z}{\mathrm{Re}\,z}$ and $\mathrm{Im}\log\beta z = \arctan\frac{\mathrm{Im}\,z}{1-\mathrm{Re}\,z}$.

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Using the notation $z \longrightarrow e^{i\phi(z)} = z/|z| = \exp\{i \operatorname{Im}\log z\}$ for the (α-compatible) operation of taking the angular part, one has by (4.42), (4.25)

$$\mathcal{L}(z) = \sum_{i\in\mathbb{Z}_3} \Pi(\phi(\beta^i z)) \tag{4.43}$$

which shows that $\mathcal{L}(z)$ is Σ_3 anti-invariant (as $\phi(\alpha z) = -\phi(z)$ and $I(\phi)$ is odd). And $\mathcal{L}(z) = \frac{1}{4i}\sum_{\gamma\in\Sigma_3} \operatorname{sign}(\gamma)\ \mathrm{Li}(e^{2i\phi(\gamma z)})$, making a Σ_3 anti-invariant projection in $\mathcal{L}(z)$ manifest, follows[38] with $\Pi(\phi) = \frac{1}{2}I(2\phi) = \frac{1}{4i}(\mathrm{Li}(e^{2i\phi}) - \mathrm{Li}(e^{-2i\phi}))$ (cf. (4.29) for $N = 2$).

Anti-invariance of $\mathcal{L}$ (fourth argument): volume of an ideal hyperbolic tetrahedron. This approach uses a geometric interpretation (4.45). The idea is to interpret the transformation behaviour of $\mathcal{L}(z)$ (under the Σ_3 operation on z) geometrically in the following sense: $z \in \mathbf{P}^1$ is interpreted as being actually a cross ratio (cf. (C.13); the definition is normalized so that $z = cr\{\infty, 0, 1, z\}$)

$$z = cr\{z_1, z_2, z_3, z_4\} = \frac{z_1 - z_3}{z_1 - z_4} \Big/ \frac{z_2 - z_3}{z_2 - z_4} \tag{4.44}$$

of four points z_1, z_2, z_3, z_4 in $\mathbf{P}^1$ and the operation of Σ_3 as the residual effect of the original Σ_4 on the z_i (cf. (C.15), (C.16)); then $\mathcal{L}(z) \in \mathbb{R}$ is understood as a geometrical quantity which transforms under Σ_4 with the sign character (of Σ_4 which induces the corresponding character on Σ_3). For this geometrical quantity one takes the hyperbolic volume of the ideal tetrahedron in hyperbolic three space $\mathbf{H}_3$ (cf. appendix C) with vertices z_1, z_2, z_3, z_4 lying on the boundary $\mathbf{P}^1_{\mathbb{C}}$ of $\mathbf{H}_3$. This is then manifestly independent of the numbering of the vertices except that the orientation changes under odd renumberings showing the anti-invariant transformation behaviour.

Here an ideal tetrahedron is a tetrahedron Δ (bounded by geodesic faces and geodesic edges)[39] with vertices z_1, z_2, z_3, z_4 on the boundary $\mathbb{C} \cup \{\infty\}$. One has vol $\Delta = \mathcal{L}(z)$ with $z = cr\{z_1, z_2, z_3, z_4\}$ (cf. appendix C) or, equivalently[40]

$$\mathrm{vol}\ \Delta(z) = \mathcal{L}(z) \tag{4.45}$$

for an ideal tetrahedron $\Delta(z)$ with vertices $(\infty, 0, 1, z)$. As a check note that $\Delta(z)$ degenerates if one of the faces degenerates, i.e. not only for $z = 0, 1, \infty$ but also for z being on a line with 0 and 1, i.e. for z real; in all these cases (4.40) vanishes as well.

Let us give an example. The symmetric hyperbolic three-simplex Δ_{sym} (with vertices on $\mathbf{P}^1_{\mathbb{C}}$ and having all six dihedral angles equal to $\pi/3$) is in the 'circle gauge' (cf. appendix) given by the vertices $\infty, -1, -\omega^2, -\omega$, so $z = -\omega = e^{i\pi/3}$ and (cf. (3.11))

$$\gamma_i = \frac{\pi}{3} \tag{4.46}$$

[38]Note the α-anti-invariant projection in $I(\phi) = \mathrm{Im}\ \mathrm{Li}(e^{i\phi}) = \frac{1}{2i}(\mathrm{Li}(e^{i\phi}) - \mathrm{Li}(e^{-i\phi}))$ making I odd.

[39]The geodescis are vertical lines and semi-circles (in vertical planes) with endpoints in the boundary $\mathbb{C} \cup \{\infty\}$; geodesic planes are vertical planes and hemispheres (over $\mathbb{C}$ and bounded by geodesics).

[40]As the z_i can be transformed to $(\infty, 0, 1, z)$ by an element of $\mathrm{Sl}(2, \mathbb{C})$ on $\mathbf{P}^1_{\mathbb{C}}$, an isometry of $\mathbf{H}_3$.

Now the volume (C.11) of a hyperbolic three-simplex becomes maximal[41] for the symmetric case (4.46) (cf. the corresponding remark about $I(\phi)$ in subsection 4.1)

$$\mathrm{vol}(\Delta_{\mathrm{sym}}) = 3\Pi\left(\frac{\pi}{3}\right) \tag{4.47}$$

which, being equal to $\frac{3}{2}I(2\pi/3)$, equals indeed $\mathcal{L}(z) = \mathrm{Im}\,\mathrm{Li}(z) = I(\pi/3)$ by (4.27).

4.4 Linear modifications

We consider now some possible slight modifications which can occur from different prespectives but all have a somewhat similar flavour. Recall that we found (4.22) for the formal triality symmetric (anti-invariant) modification $\tilde{W}$ of W

$$\tilde{W}(z) = R(z) \ \pm \frac{i\pi}{6}(\log z - \log \beta z) - \frac{\pi^2}{6}\,. \tag{4.48}$$

So $\tilde{W}$ was a less than quadratic modification of[42] R, i.e. up to a constant just a *linear* modification. We want to point here to other occurrences of such linear modifications.

For this let us recall the differential equation (4.10) for the superpotential *before* doing the holomorphic completion (which the aim to get a proper superpotential suggested)

$$\frac{dW}{d\log u} = \mathrm{Im}\log\beta u = \frac{1}{2i}(\log\beta u - \log\beta\bar{u}) \tag{4.49}$$

(where we absorbed a factor i into W). Considering z and $\bar{z}$ as two degrees of freedom like Re z and Im z, the antiholomorphic part of the r.h.s. of (4.49) is independent of the differentiation variable and one finds by giving up the holomorphicity demand on W a superpotential W_{anom} with a holomorphic anomaly

$$W_{anom}(u) = \frac{1}{2i}\,\mathrm{Li}(u) - \frac{1}{2i}\log\beta\bar{u}\,\log u \tag{4.50}$$

which is a linear modification in the holomorphic coordinate (of course quadratic when considered non-holomorphically). For easier comparison we display some relations

$$\mathrm{Re}\,W_{anom}(u) = \frac{1}{2}\,\mathrm{Im}\,\mathrm{Li}(u) - \frac{1}{2}(\mathrm{Re}\log\beta u\,\mathrm{Im}\log u - \mathrm{Im}\log\beta u\,\mathrm{Re}\log u) \tag{4.51}$$

$$\mathrm{Im}\,W_{anom}(u) = -\frac{1}{2}\,\mathrm{Re}\,\mathrm{Li}(u) + \frac{1}{2}(\mathrm{Re}\log\beta u\,\mathrm{Re}\log u + \mathrm{Im}\log\beta u\,\mathrm{Im}\log u) \tag{4.52}$$

$$\frac{1}{2}\mathcal{L}(u) = \frac{1}{2}\,\mathrm{Im}\,\mathrm{Li}(u) - \frac{1}{2}\,\mathrm{Im}\log\beta u\,\mathrm{Re}\log u\,. \tag{4.53}$$

Further, allowing [14] in (the r.h.s. of) the differential equation (4.10) of W for an additional additive constant $-\log\beta u_*$, one finds again a similar modification but with u_* constant

$$W^{\mathrm{var}}(u) = \mathrm{Li}(u) - \log\beta u_*\log u\,. \tag{4.54}$$

[41] In $\mathbf{H}_2$ $area(\Delta_2) = \pi - \sum\alpha_i$ becomes maximal ($=\pi$) for $\alpha_i = 0$ (like for the fundamental domain).

[42] In many respects (cf. section 6) the Rogers modification $R(z) = \mathrm{Li}(z) - \frac{1}{2}\log\beta z\log z$ (cf. (B.15)), which itself may be described as a quadratic (in the log's of z and its $\mathbb{Z}_3$ transforms) modification of Li, is a conceptually more natural object to consider.

Finally, including codimension four singularities like in $\mathbb{R}^4 \times \mathbf{S}^3 \times \mathbb{R}^4/\mathbb{Z}_N$ leading to non-abelian gauge symmetry on $\mathbb{R}^4 \times \mathbf{S}^3$ (cf. section 5) one has for the full superpotential[43]

$$W_{\mathrm{YM,mem}} = c\,W(u_{1,k}) + S\underline{\Phi} = cN\ \mathrm{Li}(e^{2\pi ik/N}\underline{u}^{1/N}) - iS\log\underline{u}\,. \tag{4.55}$$

4.5 Comparison with a flux superpotential

We now want to compare the membrane instanton superpotential $W \sim \mathrm{Li}$ with a flux induced superpotential[44] $W_G = \int_{W_7} G \wedge (C + i\Upsilon)$. A non-trivial G-flux turned on (as classical background) will break supersymmetry [10, 11]. So the mentioned comparison is possible only because of the absence (4.12) of proper susy vacua, i.e. one can have a non-trivial G-flux just[22] for all $u \neq 0$. One tries to *choose*[49] $\int_B G := \log \beta u$ (mimicking the quantum vev). Recall that the notion of superpotential was somewhat improper because of the non-compactness of X_7; similarly one does not have a proper Kähler potential[45] in the infinite volume case, or a flux with support on a closed cycle (here B has effectively the boundary[46] D). In the end all of this should be embedded in compact G_2 holonomy manifolds. But at least it is suggestive to see how the form of the membrane instanton superpotential may reappear here. Let us recall first the flux superpotentials [5, 9, 10].

One has, schematically, the flux-generated superpotential in type IIB on a Calabi-Yau

$$W_H = \int_{CY} H \wedge \Omega\,, \quad V_H = \int_{CY} H \wedge *H \qquad (+n) \tag{4.56}$$

(with holomorphic three-form Ω and $H = H_3^R + \tau H_3^{NS}$, cf. [5]) the associated scalar potential $V_H = \int d^2\theta\, W_H$ can be obtained from a Kaluza-Klein reduction of the kinetic term H^2 (including a topological integer $n \sim \int_{CY} H^{NS} \wedge H^R$). Similarly one has, schematically, on a G_2 holonomy manifold X with covariant constant three-form Υ [10, 6]

$$W_G = \int_X G \wedge (C + i\Upsilon)\,, \qquad V_G = \int_X G \wedge *G + (\int_X G \wedge C)^2 \tag{4.57}$$

and gains[47] in the scalar potential the Kaluza-Klein reduction of the kinetic term G^2. V_G contains, now in our non-compact case[48] X_7, a term of the form considered in (4.7)

$$\int_X G \wedge *G = \int_{B_i} G \int_{Q_i} *G \leftrightarrow^{49} \int_{D_i} C \int_{Q_i} *G \tag{4.58}$$

[43] Cf. footn. 5.10; here S is the superfield $\mathrm{tr}\, W^\alpha W_\alpha$ of highest component $\int d^2\theta\, S = \mathrm{tr}(F^2 + iF \wedge F)$.

[44] Strictly speaking this may concern in general a 'dual' G_2 holonomy manifold; the difference may concern in our case of the M-theory conifold just a transition to a phase with the role of $\mathbf{S}^3$'s exchanged.

[45] Actually W is a section of a line bundle L of $c_1(L) = \frac{1}{2\pi i}\partial\bar\partial K$ over the moduli space $\mathcal{M}$.

[46] Compactification of X gives the closed cone of boundary $Y = \mathbf{S}^3_Q \times \mathbf{S}^3_D$; for M-theory on manifolds with boundary [28] *open* membrane instanton contributions to the superpotential become important [12].

[47] Precise normalizations [6] give $\mathrm{vol}(X) = \frac{1}{7}\int \Upsilon \wedge *\Upsilon$, $\theta = \frac{1}{4\pi}\int G \wedge C \in \mathbb{R}/2\pi\mathbb{Z}$, $e^K = \frac{(2\pi)^3}{\mathrm{vol}(X)^3}$, $W_G = \frac{1}{8\pi^2}\int G \wedge (\frac{1}{2}C + i\Upsilon) \in \mathbb{C}/\frac{1}{2}\mathbb{Z}$, $V_G \sim e^K(\mathrm{vol}(X)\int |G|^2 + (2\pi\theta)^2)$ (kinetic term $e^{K/3}|\partial_i W|^2 \sim \int |G|^2$).

[48] The two effective supersymmetry transformations of the double fermionic integration in $V = \int d^2\theta\, W$ lead for u [2] from the volume (metric) in $\int_Q \Upsilon \sim \int_Q d^3x\sqrt{g}$ (Q is supersymmetric) first to the gravitino ψ and then again to the bosonic field C, more precisely to $\int_{Q_i} *G$; so symbolically one gets from the $\int G \wedge \Upsilon$ in W_G the $\int G \wedge *G$ in V_G via $\int d^2\theta \int_Q \Upsilon \sim \int_Q *G$; just as with Ω and $*H$ for a Calabi-Yau.

(at least in a schematic product ansatz where also $\theta \sim \int_{B_i} G \int_{Q_i} C \leftrightarrow^{49} \int_{D_i} C \int_{Q_i} C$ so one has still $V_G \sim \int_{D_i} C$). Now we want to argue for the P_i as representing supersymmetric vacua. For this note that the relations $u_i = 0$ and $\int_{D_i} C = 0$ (which hold at the semiclassical end P_i where $r_0 \approx \infty$) give $W|_{P_i} = 0$ and $\partial W|_{P_i} = 0$: the first from $\int_B G = \log \beta u \to 0$, the latter by $V \sim |\partial W|^2$ (at P_i where $W|_{P_i} = 0$) together with $V_G \sim \int_{D_i} C$ by (4.58) (conversely (4.7), (4.58) would give $P \neq P_i \Longrightarrow \int_D C \neq 0 \Longrightarrow \partial W \neq 0$)

$$P = P_i \Longrightarrow W|_P = 0 = \partial W|_P \tag{4.59}$$

Note further that (specialising to our non-compact X_7)

$$\frac{\partial W_G}{\partial \Phi_i} = \int_X G \wedge \delta_{B_i} = \int_{B_i} G \tag{4.60}$$

suggesting[49] with (3.7) again the differential equation (4.10) (and (4.59) by (4.12))

$$\frac{dW_G}{d \log u_i} = \log \beta u_i \, . \tag{4.61}$$

Remark: there is a formal similarity between two invariance-adjustments of the superpotential $W_G = -i$ Li. The adjusted imaginary part $\mathcal{L}$, depending on $u = e^{i\Phi}$, is invariant under monodromy from the $\pi_1(\mathbf{P}^1_u \backslash \{0, 1, \infty\})$ action (with *additive* shifts $\Delta_{l_0}\Phi = 2\pi$, $\Delta_{l_1} W_G = 2\pi i \Phi$); a term like DW (now dependent on Φ) is invariant under Kähler transformations (with *multiplicative* shift $W \to W e^{-F}$ and $K \to K + F + \bar{F}$; so the section W is adjusted to a well-defined function in $e^G = e^K |W|^2$). From the general relation $\partial_i K = \frac{i}{2} \frac{1}{\mathrm{vol}(X)} \int \chi_i \wedge * \Upsilon$ (where $\Upsilon = \mathrm{Im}\, \Phi^i \, \chi_i$ is the harmonic decomposition) [6] one gets in our local situation of X_7 with one 3-cycle Q, $\chi = \delta_B$ and $\mathrm{vol}(X) = \mathrm{Im}\, \Phi \int_B * \Upsilon$ the finite expression $\partial_i K = \frac{i}{2} \frac{1}{\mathrm{vol}(X)} \int_B * \Upsilon = \frac{i}{2} \frac{1}{\mathrm{Im}\, \Phi_i}$, which gives for the covariant derivative $DW = \partial W + \frac{i}{2} \frac{1}{\mathrm{Im}\, \Phi_i} W$ (so that the difference to the ordinary derivative ∂W vanishes when approaching via vol $Q \to \infty$ the end). So one can compare (with $\mathrm{Li}(u = e^{i\Phi})$)

$$\mathrm{Im}\, \Phi \cdot \mathrm{Im}\, DW_G = \frac{1}{2} \mathrm{Im}(iW_G) + \mathrm{Im}\, \Phi \, \mathrm{Im} \frac{\partial (iW_G)}{\partial (i\Phi)} \tag{4.62}$$

$$\mathcal{L} = \mathrm{Im}\, \mathrm{Li} + \mathrm{Im}\, \Phi \, \mathrm{Im} \frac{\partial \, \mathrm{Li}}{\partial (i\Phi)} \tag{4.63}$$

which leads, up to the factor 1/2, with $iW_G = \mathrm{Li}$ to a certain formal parallelism.

5. Codimension 4 singularities

When $X = \mathbb{R}^4 \times \mathbf{S}^3$ is divided through by a finite subgroup Γ of SU(2) one obtains $X_{1,\Gamma} \cong \mathbb{R}^4/\Gamma \times \mathbf{S}^3$ and $X_{2,\Gamma} \cong X_{3,\Gamma} \cong \mathbf{S}^3/\Gamma \times \mathbb{R}^4$ leading as effective four-dimensional field theories to an ADE gauge theory and to a theory without massless fields, respectively (the latter explaining the conjectured mass gap of the former) (Γ operates always on the first factor of $\mathrm{SU}(2)^3/\,\mathrm{SU}(2)_D$, the i in X_i denotes which factor is filled in).

[49] In the compact case note that G-flux, on a *compact* $K3$ fibre, say, of a $K3$ fibered W_7, is quantised (in units of 2π over tension), so constant over the moduli space (and the duality [29] with the heterotic string might then be obstructed as for type IIA [30])). Here $\int_B G = \int_D C$, being zero classically, becomes in the quantum domain the varying expression Im $\log \eta$, which may now be *mimicked* (!) by prescribing for each u a corresponding classical flux background (which entails the ensuing formal similarities).

Relations between the observables given by the η_i-variables. In more detail [2] let $N = |\Gamma|$ and $Y_\Gamma = \Gamma\backslash \mathbf{S}^3 \times \mathbf{S}^3 = \Gamma\backslash \mathrm{SU}(2)^3/\mathrm{SU}(2)$ where Γ acts on the *first* factor. Clearly the triality symmetry Σ_3 is broken down to $\mathbb{Z}_2$. The three-cycles D_i project to the $D_i^{'}$ with $D_1 \simeq ND_1^{'}$ and $D_i \simeq D_i^{'}$ for $i > 1$ and one has $ND_1^{'} + D_2^{'} + D_3^{'} = 0$ and $N\alpha_1^{'} + \alpha_2^{'} + \alpha_3^{'} = N\pi$ (with $\alpha_i^{'} = \int_{D_i^{'}} C$) from the consideration of the membrane anomaly. The ensuing relation $\eta_1^N \eta_2 \eta_3 = (-1)^N$ for the variables $\eta_1 = \exp\{k\frac{2f_3+f_1}{3N} + i\alpha_1^{'}\}$ and $\eta_i = \exp\{k\frac{2f_{i-1}+f_i}{3} + i\alpha_i^{'}\}$ for $i > 1$ entails that now the η_i are not a simple $\mathbb{Z}_3$ orbit with $\eta_3 = \beta\eta_1, \eta_2 = \beta^2\eta_1$, as in (3.4), but that rather

$$A\text{-series} \qquad \eta_2 = (\beta^2\eta_1)^N\,, \quad \eta_3 = (\beta\eta_1)^N\,. \tag{5.1}$$

This is actually only the case if Γ is just a cyclic group, corresponding to the A-series (in the type-IIA reinterpretation this means that one has wrapped N $D6$-branes on the $\mathbf{S}^3$ respectively has N units of Ramond flux on the $\mathbf{S}^2$). For the two different types of D_n singularity in M-theory ($n \geq 4$), with gauge group $\mathrm{SO}(2n)$ and $\mathrm{Sp}(n-4)$ (of dual Coxeter numbers $h = 2n-2$ and $h' = n-3$, fulfilling $N = h + 2h'$), respectively, where the latter 'exotic' case leads to the new semiclassical limit point $\eta_1 = -1$ (beyond $0, 1, \infty$) of the quantum modulio space $\mathcal{N}_\Gamma$, the relations are

$$D\text{-series} \qquad \eta_2 = \left(\beta^2\eta_1\right)^h \left(\beta^2(-\eta_1)\right)^{2h'}\,, \quad \eta_3 = (\beta\eta_1)^h\,(\beta(-\eta_1))^{2h'}\,. \tag{5.2}$$

Finally for the E-series one has (with $\omega_t = e^{2\pi i/t}$) new classical limits at $\eta_1 = \omega_t^\mu$: the different E-singularities in M-theory are indexed by an integer t dividing some of the Dynkin indices k_i of the E-group and an integer μ with $1 \leq \mu < t$ and $(\mu, t) = 1$ (for $t \geq 2$; for $t = 1$ is $\mu = 0$). With $h_t = \frac{1}{t}\sum_{t|k_i} k_i$ the dual Coxeter number of the gauge group K_t in M-theory at a G-singularity of index t one has (in general)

$$E\text{-series} \qquad \eta_2 = \prod_{t,\mu}\left(\beta^2\omega_t^\mu\eta_1\right)^{t\,h_t}\,, \quad \eta_3 = \prod_{t,\mu}\left(\beta\omega_t^\mu\eta_1\right)^{t\,h_t}\,. \tag{5.3}$$

Relation to the instanton expansion parameters u_i. After the mutual η_i relations let us also give the analogues of the relation (3.7) between the η_i and the instanton parameters u_i. Let us restrict us to the A-series (in general a membrane wrapped on $\mathbf{S}^3_Q \subset X_{1,\Gamma}$ corresponds to t instantons in K_t.). The different $X_{i,\Gamma}$ are defined by the 'filling in' condition $D_i^{'} \simeq 0$. Consider the two cases $i = 1$ resp. $i > 1$ separately. At the center of $X_{1,\Gamma} = \mathbb{R}^4/\Gamma \times \mathbf{S}^3$ lies $\mathbf{S}^3 = Q_1^{'} \simeq \pm D_{i>1}^{'}$ (as the membrane instanton corresponds in the four-dimensional supersymmetric $\mathrm{SU}(N)$ gauge theory to a point-like Yang-Mills instanton, note that, because of chiral symmetry breaking as detected by the gluino condensate, the local parameter at P_1 is not $u_1 = \exp\{i(\int_{Q_1^{'}} C + i\Upsilon)\}$ but rather $u_1^{1/N}$). As earlier one gets $u_1 = \eta_2$ or with (5.1)

$$\beta\left(u_1^{1/N}\right) = \eta_1 \tag{5.4}$$

(again footn.'s 11, 13 apply). For $i > 1$ one has at the center of $X_{i,\Gamma} = \mathbb{R}^4 \times \mathbf{S}^3/\Gamma$ lying $\mathbf{S}^3/\Gamma = Q_i^{'} = \pm D_1^{'}$ (this time u_i is a good local parameter at P_i). $u_3 = \eta_1$ leads now to

$$\beta u_3 = \eta_3^{1/N}\,. \tag{5.5}$$

Superpotential. Let us consider the ensuing superpotential evaluations (4.10) (again for the A-series). On $X_{i,\Gamma}$, where $i = 1$ or 3, one gets for $dW/d\log u_i$ now $\int_{\mathbf{S}^3/\Gamma} C = \operatorname{Im}\log\eta_1$ and $\int_{\mathbf{S}^3} C = \operatorname{Im}\log\eta_3$, respectively, and finds (after holomorphic completion) with (5.4), (5.5)$(k \in \mathbb{Z}_N)$

$$W(u_{1,k}) = \int_0^{u_1} \log\beta(t^{1/N})d\log t = N\,\mathrm{Li}(\omega_N^k u_1^{1/N}) \tag{5.6}$$

$$W(u_3) = \int_0^{u_3} \log(\beta t)^N d\log t = N\,\mathrm{Li}(u_3) \tag{5.7}$$

($u^{1/N}$ principal value, $\omega_N = e^{\frac{2\pi i}{N}}$). $u_1^{1/N} = \beta^2 u_3$ by $u_1 = \eta_2$, $u_3 = \eta_1$ and (5.1). By (4.29)

$$\sum_{k\in\mathbb{Z}_N} W(u_{1,k}) = \mathrm{Li}(u_1) \tag{5.8}$$

$$W(u_3) = N\,\mathrm{Li}(u_3)\,. \tag{5.9}$$

Now consider the four-dimensional interaction $\operatorname{Im}\int_{\mathbb{R}^4} d^4y \int d^2\theta S\underline{\Phi}$ with $S = \operatorname{tr} W^\alpha W_\alpha$ the 'glueball' chiral superfield of lowest component the gaugino bilinear $\operatorname{tr}\lambda^2$ and highest component $\int d^2\theta \operatorname{tr} W^2 = F^2 + iF\wedge F$ (W^α the field-strength superfield of highest component $F + i *_4 F$) and $\underline{\Phi}$ the superfield of lowest component $\Phi = \int_Q C + i\Upsilon$ and highest component $\int_Q *_{(X_7)}G$ by (4.6). Just as the coupling constant in front of the kinetic term of the seven-dimensional gauge fields on $\mathbb{R}^4 \times \mathbf{S}^3$ gets rescaled by $\mathrm{vol}(\mathbf{S}^3) = \operatorname{Im}\Phi$ in four dimensions, one has an interaction $\int_{\mathbb{R}^4\times\mathbf{S}^3} \operatorname{tr} F\wedge F\wedge C$ (gauge theory instantons carry membrane charge) so $\operatorname{Re}\Phi$ leads to the four-dimensional theta-angle θ. On the other hand one has the interaction $\int_{\mathbb{R}^4\times\mathbf{S}^3} d^4y \operatorname{tr}\lambda^2 \wedge *_{(X_7)}G$ and as $\operatorname{tr}\lambda^2$ gets a vev $\sim \Lambda^3 e^{\frac{i}{N}\theta}\omega_N^k$ ($k \in \mathbb{Z}_N$, the N different vacua from chiral symmetry breaking) one finds [2] again (cf. (4.6)) the interaction proportional to $\int_{\mathbf{S}^3} *G$.

Taken together (with relative weight factor[50] $c \sim e^{-1/g^2_{\mathrm{YM}}} e^{\frac{i}{N}\theta}$) with the membrane instanton contribution u one finds as superpotential $W_{\mathrm{YM,mem}^{(1)}} = S\underline{\Phi} + N^2 c u_1^{1/N}$ of critical point $S = -iNcu_1^{1/N}$. This gives $\Phi = -i\log(\frac{i}{Nc}S)^N$ and $W_{\mathrm{eff}}(S) = -iS\log(\frac{i}{Nc}S)^N + iNS$ of critical point $(\frac{i}{Nc}S)^N = 1$ or $\operatorname{tr} W^2 = -iNc\omega_N^k$. We will consider elsewhere the critical point and the effective superpotential for the full

$$W_{\mathrm{YM,mem}} = S\underline{\Phi} + Nc\,W(u_{1,k})\,. \tag{5.10}$$

Remark: there is another $\mathbb{Z}_N$ relation (5.11) besides (5.8) (i.e. (4.29)) which would be interesting to relate with the $\mathbb{Z}_N$ of $\mathrm{SU}(N)$ or to provide a gauge-theoretic meaning.

The expression $R(z) = \frac{1}{2}\Big(\mathrm{Li}(z) - \mathrm{Li}(1-z)\Big) + \frac{\pi^2}{12}$, cf. (B.15), is more suitable for expressing some Li relations. Actually one has a relation (cf. appendix, section B)

$$\sum_{l=1}^{N-1} R\left(\frac{\sin^2\frac{\pi}{N}}{\sin^2 l\frac{\pi}{N}}\right) = R(1) + \sum_{i=1}^{N-2} R\left(\frac{\sin^2\frac{\pi}{N}}{\sin^2(i+1)\frac{\pi}{N}}\right) = \frac{\pi^2}{6}\left(1 + \frac{3(N-2)}{N}\right). \tag{5.11}$$

[50] By a shift $\Phi \to \Phi + \Phi_0$ the c can be identified with a shift in the bare coupling constant, so $c \sim e^{\frac{i}{N}\Phi_0}$; further there is an order N^2 factor [33, 34].

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Here the argument is $1/Q_{i0}^2$ (where $\mathbf{i+1} = Sym^i\,\mathbf{2}$ with action $\mathrm{diag}(z^i, z^{i-2}, \ldots, z^{-i})$)

$$Q_{i0} = \frac{\sin(i+1)\frac{\pi}{N}}{\sin\frac{\pi}{N}} = \left|\frac{1-\omega_N^{i+1}}{1-\omega_N}\right| = \mathrm{tr}_{\mathbf{i+1}}\,\omega_N \tag{5.12}$$

with $\mathbb{Z}_N \hookrightarrow \mathrm{Sl}(2,\mathbb{C})$ via $\mathrm{diag}(z, z^{-1})$ for $z = \omega_N$. Eq. (5.11) is interpretable [43] as the evaluation (cf. (6.14)) of a Cheeger Chern-Simons class on a generator of $H_3(\mathbb{Z}_N, \mathbb{Z})$: consider the embedding $\mathbb{Z}_N \hookrightarrow \mathrm{PSl}(2,\mathbb{R})$ given by

$$\begin{pmatrix} \cos\frac{\pi}{N} & -\sin\frac{\pi}{N} \\ \sin\frac{\pi}{N} & \cos\frac{\pi}{N} \end{pmatrix}. \tag{5.13}$$

Furthermore one has a geometrical interpretation [44] that $\sum_{j=1}^k R(t_j) = \frac{\pi^2}{6}n$ for a certain integer n when a 3-manifold M is triangulated by k oriented tetrahedra T_j. Here for each vertex i ($i = 1, \ldots, N$) a real number x_i is given and one has associated to the tetrahedron T_j the corresponding cross ratio $t_j = cr\{x_a, x_b, x_c, x_d\}$. As the set of tetrahedra forms a triangulation the boundary $\partial \sum_{j=1}^k T_j = 0$ of the associated 3-chain is zero and this implies the relation $\sum_{j=1}^k t_j \wedge (1-t_j) = 0$ which then implies (5.11). (cf. appendix D)

6. Interpretation and outlook

The preceding investigations cause three sorts of questions. First, one may dwell on some of the points touched already: the identification of the relevant coordinates (u_i, η_i) (cf. discussion around (3.7)), especially the globalization question with possibly a direct connection to the type-IIA approach [14] which uses special flat coordinates of $N=2$; the question of holomorphic completion of the superpotential in $\frac{\partial W}{\partial \log u} = \mathrm{Im}\log\beta u \to \log\beta u$, respectively the holomorphy violation (by the boundary; cf. the E_2 anomaly, even in superpotential contexts); the interpretation of the transformation rules of W (section of Heisenberg bundle, or even a balancing argument as in (4.17); W_G monodromy from the C field shifts); also to see directly, before evaluation, the connection $X_7 \to \Delta(z)$ (and that a G flux (?) evaluates W_G on X_7 to the (complexified, cf. below) invariant $\mathrm{vol}(\Delta)$).

The second type of questions concerns an interpretation of the results obtained so far (section 6.1). Finally their posssible extension to more generic (compact) cases and placement in a greater conceptual context (sketched in the more speculative section 6.2, described in more dateil elsewhere [41]). (For relations to type-IIA string theory cf. [35].)

6.1 Local interpretation

The universal object over the quantum moduli space. We now want to compare the structures found with corresponding constructions in the description of pure $N=2$ SU(2) gauge theory as given in the Seiberg/Witten (S/W) set-up [25]. There one was interested in the section $\begin{pmatrix} \partial_u a_D \\ \partial_u a \end{pmatrix} = \begin{pmatrix} \int_\beta \partial_u \lambda \\ \int_\alpha \partial_u \lambda \end{pmatrix} = \begin{pmatrix} \int_{C^3_\beta} \Omega \\ \int_{C^3_\alpha} \Omega \end{pmatrix}$ of the flat bundle given by the first cohomology of the universal elliptic curve $\mathcal{E} \to \mathbf{P}^1_u$ over the quantum moduli space $\mathbf{P}^1_u = \Gamma(2)\backslash\mathbf{H}_2$; the fibre over u is $H^1(E_u, \mathbb{C})$ with the elliptic curve E_u given as two-fold covering of $\mathbf{P}^1$ branched at $\infty, 0, 1, u$ (degenerating for u being one of the three points $\infty, 0, 1$; encircling the corresponding singularities gives monodromy elements generating $\Gamma(2)$).

Recall the relation[51] of the (meromorphic) periods of the S/W curve $\mathcal{C}$ (describing a gauge theory engineered on a $K3$-fibered Calabi-Yau X in type IIA) and the (holomorphic) periods of the mirror Calabi-Yau W; one has with cycles $C_3 \subset W$, $C_1 \subset \mathcal{C}$

$$\int_{C_3} \Omega \sim \int_{C_1} \lambda\,. \tag{6.1}$$

Now replace this universal family of elliptic curves by a family of three-manifolds, or rather (in our local case) three-simplices: we associate to $z \in \mathbf{P}^1_z$ the hyperbolic geodesic three-simplex given by the ideal tetrahedron $\Delta(z)$ in[52] $\mathbf{H}_3$ with vertices $\infty, 0, 1, z$; again the construction degenerates for z being one of the three points $\infty, 0, 1$.

So the quantum regime of the universal local structure provided by the non-compact M-theory conifold X_7 (of quantum moduli space $\mathbf{P}^1_{\mathbb{C}}$) corresponds to the variation of $\Delta(z)$ over $z \in \mathbf{P}^1_{\mathbb{C}} = \partial\mathbf{H}_3$ (cf. vol $\Delta(z) = \mathcal{L}(z)$ in (4.45)).

We will compare to corresponding expressions in our set-up the pair a, a_D and the Kähler potential (in the S/W set-up) as relations of dual torus periods ($\mathcal{F}$ prepotential)

$$a_D = \frac{\partial \mathcal{F}}{\partial a}\,, \qquad K = -\operatorname{Im} a\bar{a}_D \tag{6.2}$$

$$a = \oint_\alpha \lambda\,, \qquad a_D = \oint_\beta \lambda\,. \tag{6.3}$$

The quantum coordinate is not a^2 but rather the corrected (cf. remark 3 below) quantity

$$\frac{1}{2\pi i} u = \frac{1}{8}\left(\mathcal{F} - \frac{1}{2} a \partial_a \mathcal{F}\right) \tag{6.4}$$

and in the stringy realization the quantum coordinate u becomes purely geometrical[58,53]

$$u = \Xi^2 = \int_{\mathcal{C}_3} \Omega \tag{6.5}$$

which is made possible by going to the mirror description in type IIB.

With the modification R of Li (cf. (B.15)), whose relevance will emerge below repeatedly, we can compare to (6.4) as one has (so $a, \mathcal{F}, u$ are related to[54] $i\Phi = \log z, \mathrm{Li}, R$)

$$R = \mathrm{Li} - \frac{1}{2} \log z\, \partial_{\log z} \mathrm{Li}\,. \tag{6.6}$$

[51]Where $\mathcal{C}$ is built up as a covering over $\mathbf{P}^1$ 'the same way' (replacing the intersection lattice $H^2(K3, \mathbb{Z})$ by a zero-dimensional spectral set) that W is built up as $K3$ fibration over $\mathbf{P}^1$.

[52]Think of a different copy of $\mathbf{H}_3$ over each point $z \in \mathbf{P}^1_z$ as ambient space for $\Delta(z)$ just as one thinks of a different copy of the Weierstrass embedding plane $\mathbf{P}^2_{x,y,z}$ over each $u \in \mathbf{P}^1_u$ as ambient space for E_u so that in both cases one really ends up with a fibration (where the fibres are disjoint).

[53]The prepotential $F(X^0, X^1, \ldots, X^n)$ of the periods X^i is related to the prepotential $\mathcal{F}(t^1, \ldots, t^n)$ of the Kähler coordinates $t^A = X^A/X^0$ via $F = (X^0)^2\mathcal{F}$, giving the relation $\frac{1}{2}(X^0)^{-1}\partial_{X^0} F = \mathcal{F} - \frac{1}{2} t^a \partial_a \mathcal{F}$. The period a_D is related to the conifold via $\Xi_6 \sim a_D \sim x_+ \sim \tilde{u} - 1$ with Ξ_6 the period related to the 6-cycle in type IIA, respectively the vanishing $\mathbf{S}^3$ of the conifold in type IIB. The type-IIA perspective on the conifold, the period a_D and its relation to the dilogarithm are discussed further in [35].

[54]We call here the u modulus of (2.14) z to avoid mix-up with the S/W u.

So the distinctive feature of the S/W solution, that a quantity in the bulk of the quantum moduli space has a purely geometric expression like the mentioned periods (typical for string dualities), resembles the way how in our $N = 1$ set-up the quantum corrected quantity $W(u)$ becomes an integral of classical geometry on a 'dual object' (lying over a respective point in the quantum moduli space), i.e. $\mathcal{L} = \int_\Delta vol$ (respectively its complexified extensions described below which suggest the whole point of view).

Remarks.

1. The membrane anomaly becomes manifest in the described global quantum model

$$\sum_{\mathbb{Z}_3} \int_{\mathbf{S}^3_{D_i}} C = \pi \longleftrightarrow \sum_{\mathbb{Z}_3} \mathrm{tors}(\gamma_i) = \pi \tag{6.7}$$

   (using $\mathbf{S}^3_Q$ for the $\mathbf{S}^3_D$ (phases)[13]). This points to a connection[55] between the direct manifestation (C.10) of the anomaly in the dual hyperbolic model and the proof (2.18).

2. The different *three-dimensional* structures we encounter (membrane instantons and the modulus $\Phi = \int_Q C + i\Upsilon$; the Heisenberg bundle $\underline{\mathcal{H}}$ as built up from Li, log and 1; the (solid) tetrahedron $\Delta(z)$ and its volume[56]) have corresponding two-dimensional structures in the S/W set-up (world-sheet instantons; $H^1(E_u)$ as built up by (a_D, a); E_u).

3. In the S/W set-up there is also the relation with a flux superpotential which we contemplated for our case in subsection 4.5. For this recall the stringy realization of the $N = 2 \to N = 1$ mass breaking. According to [5] the quantum corrected version $W = mu = m\langle \mathrm{tr}\, \Phi^2 \rangle$ of the classical ($u \approx a^2/2$) mass deformation in the field theory[57] is realised as a flux induced superpotential $W = \int_W \Omega \wedge H_3$ in the type-IIB string, essentially because u occurs among the Calabi-Yau periods (cf. below). The stringy realization proposed in [5] of this scenario started from the type-IIA superpotential

$$W_{\text{flux}} \sim \int_X H_2 \wedge t \wedge t \sim \left(\int_{\mathbf{P}^1_b} H_2 \right) \cdot \mathrm{vol}(K3) = n_{\text{flux}}\, \partial_s \mathcal{F} \tag{6.8}$$

   where[58] the entries S, $\partial\mathcal{F}/\partial S$ of the type-IIA period vector correspond to $\mathrm{vol}(\mathbf{P}^1_b)$, $\mathrm{vol}(K3)$.

4. Having emphasized analogies between X_7 and the S/W set-up let us point also to a difference. In the S/W set-up at the three special points $u = \infty, +1, -1$ BPS

[55] One may look at an anlogue of $\mathbf{P}^2_H \to \Delta$, a $T^2 \cong \mathcal{U}(1)^3/\mathcal{U}(1)_D$ fibration $\mathbf{P}^2_{\mathbb{C}} \to \Delta$ (cf. [36]).

[56] Or some hyperbolic 3-manifold M_3 with its volume and Chern-Simons form $\mathcal{C}^{CS}$, cf. below.

[57] Giving mass to the chiral multiplet Φ of the vector multiplet, and so the breaking $N = 2 \to N = 1$. As near $\tilde{u} = \pm 1$ a monopole and a dyon become massless one gets by including the light states $W = mu + (a_D - a_0)\phi\tilde{\phi}$ which leads to monopole condensation and locking on $\tilde{u} = \pm 1 \longleftrightarrow a_D = a_0$.

[58] Actually analytic continuation shows that this expectation has to be refined [27]: $W = mu$ is then given by $W = mu \sim 2i\Xi^2_\infty + \Xi^4_\infty = 2it + \partial_s \mathcal{F}$.

states become massless, the W-boson, a monopole and a dyon, respectively. Being BPS states, in the string theory embedding the relation between mass and volume is saturated, the respective cycles of homology classes N, N^+, N^- shrink at the special points and fulfill

$$N = N^+ + N^- \,. \tag{6.9}$$

Note that not all of the three special points are on the same footing but some of them ($u = \pm 1$) are more equal (in the stringy representation in type II these correspond to $\mathbf{S}^3$'s in the mirror Calabi-Yau whereas $u = \infty$ corresponds to $\mathbf{S}^2 \times \mathbf{S}^1$, indicating hypermultiplets and a vector multiplet, respectively. In the type-II conifold transition the two small resolutions are also more equal, i.e. the type-II reduction breaks Σ_3 to $\mathbb{Z}_2$). These two points $u = \pm 1$ lie on the curve of marginal stability. The potential decay of BPS states when crossing such a curve were considered in investigations about singularities of special lagrangian three-cycles [37] from the perspective of transitions that occur for corresponding supersymmetric three-cycles in the Calabi-Yau manifold. In [37] two different types of such singularities are considered, modelled (in $\mathbf{C^3}$) respectively on a T^2-cone and two real 3-planes. The latter case was exemplified above in (6.9) and considered also in [38, 39] (and [2] when considering the cone over $\mathbf{P}^3$) whereas the former is related to the case considered in [14, 2] (the case of the cone over $\mathbf{S}^3 \times \mathbf{S}^3$) and the present paper. Here the corresponding relation between the homology classes of the three respective cycles which become nullhomologous at the three special points is

$$D_1 + D_2 + D_3 = 0 \,. \tag{6.10}$$

6.2 Global interpretation

Compact G_2 holonomy manifolds. Now consider compact G_2 holonomy manifolds X_7, $K3$ fibered over $\mathbf{S}^3$ (replacing the previous local fibre $K3^{\mathrm{decomp}} = \mathbb{R}^4$), with singularities not just of codimension 7 (and potentially 4) but also codimension 6. The latter case where the discriminant in the base $\mathbf{S}^3$ of X_7 is of codimension two, the 'discriminant link' $l = \cup_j^h \gamma_j$ (a union of h circles), will be especially relevant to make the connection to the hyperbolic 3-manifold M_3.

More precisely, we will be concerned on the one hand with the case of a codimension 7 singularity of the classical geometry locally modelled after the cone over $Y = \mathbf{S}^3 \times \mathbf{S}^3$, or even a situation with many, say h_X, local 'ends' modelled that way (having one $\mathbf{S}^3$ as base of the $K3$ fibration brings a certain asymmetry into the description). On the other hand codimension 7 singularities arise as the cone over $\mathbf{P}^3_{\mathbb{C}}$, the $\mathbf{S}^2$-twistor space over $\mathbf{S}^4$. Consider here a component in the discriminant link given by just an unknot γ and let $\mathcal{S} \subset \mathbf{S}^3$ be a spanning Seifert surface, not touching, say, the other link components. The total cycle traced out[59] by following the cycle $\mathbf{S}^2_{x,y,z}$ in the $K3$ fibre, which vanishes over $\gamma = \partial\mathcal{S}$, through the whole $\mathcal{S} = D^2_{v,w}$ leads to an $\mathbf{S}^4 = \{x^2 + y^2 + z^2 + v^2 + w^2 = 1\}$ contributing to $b_4(X) = b_3(X)$.

[59] Fibre singularities relate to the cohomology of a total space: for an elliptic $K3$ for example one gets $\mathbf{S}^2$'s building up $H^2(K3)$ (besides base and fibre) from paths P connecting points p, p' (of codimension 2 in the base as is our link $l \subset \mathbf{S}^3$) in the base $\mathbf{P}^1$ (so $\partial P = p' - p$) over which an $\mathbf{S}^1$ in the fibre shrinks.

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Hyperbolic 3-manifolds. Above we considered the 3-dimensional structure given by the ideal tetrahedron Δ in $\mathbf{H}_3$ and studied its volume. Actually one will consider a two-fold generalisation. One refines the volume invariant and generalises the tetrahedron to smooth manifolds. We recall an universal cohomological interpretation of the dilogarithm superpotential starting from the hyperbolic simplex volume computation related to its single-valued cousin $\mathcal{L}$ which points to the consideration of the complexified Chern-Simons invariant of hyperbolic 3-manifolds (the volume combined with the Chern-Simons invariant).

Concerning the first issue one pairs the volume with the Chern-Simons invariant as suggested by the cohomological interpretation of the occurrence of Li (or R) in the hyperbolic volume computations (cf. appendix D and below) together with the Chern-Simons reformulation of three-dimensional gravity [40]. Concerning the second issue one will consider general hyperbolic 3-manifolds M and the way the refined (complexified) Chern-Simons invariant varies over the hyperbolic deformation moduli space of M; this shows [41] (from a triangulation by simplices) how the dilogarithm occurs in this variation.

Concerning the generalisation to smooth manifolds note that just as in the case of the upper half-plane one can study now discrete torsion-free subgroups Γ of the full group $\mathrm{PSl}(2,\mathbb{C})$ of orientation-preserving isometries of $\mathbf{H}_3$ and look for the corresponding (orientable) hyperbolic three-manifold M (complete riemannian manifold of constant curvature -1 of finite volume) given by the non-compact quotient $\Gamma\backslash\mathbf{H}_3$ (any such M arises this way).[60] The geodesic simplices we studied occur, just as in the well known upper half-plane case, as (parts of) fundamental domains for suitable group actions and the sum of their volumes gives the volume of the quotient manifold.

Cohomological interpretation. For $E \to M$ a differentiable $\mathrm{Gl}(n,\mathbb{C})$ bundle with *flat* connection θ one finds from the Bockstein exact sequence that $c_2(E)$ lies in the image of the Bockstein homomorphism β

$$H^3(M,\mathbb{C}/\mathbb{Z}) \xrightarrow{\beta} H^4(M,\mathbb{Z}) \longrightarrow H^4(M,\mathbb{C})\,. \tag{6.11}$$

The second *Cheeger Chern-Simons class* (appendix D) gives a canonical choice of a preimage

$$\hat{C}_2(\theta) \in H^3(M,\mathbb{C}/\mathbb{Z})\,. \tag{6.12}$$

With ω a $\mathbb{C}$-valued $\mathrm{Sl}(2,\mathbb{C})$ invariant three-form on $\mathrm{Sl}(2,\mathbb{C})/\,\mathrm{SU}(2) = \mathbf{H}_3$ one finds a $\mathbb{C}/\mathbb{Z}$ valued Eilenberg-MacLane cochain $\mathcal{I}(\omega)$ with

$$\hat{C}_2 = \mathcal{I}(\omega)(g_1,g_2,g_3) = \int_{\Delta(z)} \omega\,. \tag{6.13}$$

One finds then[61] (with $L(z) = R(z) - \frac{\pi^2}{6}$, cf. (B.15))

$$2\,\mathrm{Re}\,\hat{C}_2 = \frac{1}{2\pi^2}L(z) \quad (\mathrm{mod}\ 1/24)\,, \qquad 2\,\mathrm{Im}\,\hat{C}_2 = \frac{1}{2\pi^2}\mathcal{L}(z) \tag{6.14}$$

[60] Cf. that a closed surface of genus $g > 1$ admits a metric of constant curvature -1 and is isometric to $\Gamma\backslash\mathbf{H}_2$. Note that by the Mostow/Prasad rigidity theorem two hyperbolic threefolds of finite volume with isomorphic fundamental groups are actually isometric, the volume is a topological invariant.

[61] With $\mathrm{Re}\,\hat{C}_2$ evaluated on $H_3(\mathrm{Sl}(2,\mathbb{R})^\delta)$ for $z \in \mathbb{R}$. One can give a similar interpration for $\mathrm{Re}\log$ and its relation to $\hat{C}_1 \in H^1(\mathrm{Gl}(\mathbb{C}),\mathbb{R})$ just as $\mathcal{L}$ represents part of $\hat{C}_2 \in H^3(\mathrm{Gl}(\mathbb{C}),\mathbb{R})$.

Transition to hyperbolic 3-manifolds. Let M now be a closed 3-manifold of hyperbolic structure given by $M \cong \Gamma_{hol}\backslash\mathbf{H}_3$ or equivalently by the holonomy representation $h : \pi_1(M) \to (P)\,\mathrm{Sl}(2,\mathbb{C})$, respectively by a flat $(P)\,\mathrm{Sl}(2,\mathbb{C})$ bundle over M; this is pulled back from the universal bundle U over the classifying space $\mathrm{BSl}(2,\mathbb{C})^\delta$ by a base map $m : M \to \mathrm{BSl}(2,\mathbb{C})^\delta$ so one can evaluate[62] $\hat{C}_2 \in H^3(\mathrm{BSl}(2,\mathbb{C})^\delta)$ (cf. appendix D) on the class in $H_3(\mathrm{BSl}(2,\mathbb{C})^\delta)$ given by M

$$\mathrm{Re}\,\hat{C}_2(M) \sim CS(M)\,, \qquad \mathrm{Im}\,\hat{C}_2(M) \sim \mathrm{vol}(M)\,. \tag{6.15}$$

So the proper cohomological interpretation of vol $\Delta(z) = \mathcal{L}(z)$ leads to the consideration of hyperbolic 3-manifolds M for which the second Cheeger Chern-Simons class is given by (6.15) with universal evaluation (6.14). For a hyperbolic 3-manifold M the invariant $\int_{M_3} \mathcal{C}^{CS} + i\,\mathrm{vol}$ is studied (in this complex Chern-Simons theory (as in 3D gravity) one has naturally the complex pairing of volume and the CS 3-form field, cf. [41]).

The hyperbolic deformation moduli space is defined via periods of the generator loops m_i, l_i (for h a suitable one-form) for the (assumed) toroidal ends

$$v_i(\mathbf{u}) = \frac{\partial \mathcal{G}}{\partial u_i}\,, \qquad K(\mathbf{u}) := 2\pi \sum_i \mathrm{length}(\gamma_i) = -\sum_i \mathrm{Im}\, u_i \bar{v}_i \tag{6.16}$$

$$u_i = \pm \int_{m_i} 2h^*\,, \qquad v_i = \pm \int_{l_i} 2h^*\,. \tag{6.17}$$

In other words there exists again a prepotential $\mathcal{G}$ and again the expression

$$f = \frac{1}{4}(2 - \mathbf{u}\partial_{\mathbf{u}})\mathcal{G} \tag{6.18}$$

has a purely geometrical description (Dehn filling the ends via solid tori will be involved)

$$f = \mathrm{vol}(M) + i\mathcal{C}^{CS}(M) = \int_M \omega\,. \tag{6.19}$$

Eqs. (6.16)–(6.19) compare to (6.2)–(6.5) ($a, \mathcal{F}, u$ relate to $\mathbf{u}, \mathcal{G}, f$). One defines invariants

$$\begin{aligned} I(M) &= \exp\left\{\int_M \frac{2}{\pi}\mathrm{vol} + i\mathcal{C}^{CS}\right\} = \exp\left\{\int_M \omega\right\} \\ \lambda(\gamma) &= \exp\{\mathrm{length}(\gamma) + i\,\mathrm{tors}(\gamma)\} = \exp\left\{\int_\gamma 2h^*\right\} \end{aligned} \tag{6.20}$$

generalising the occurrence of Li and log, or their real cousins $\mathcal{L}$ and Re log as three- resp. one-dimensional volumes (4.45), (C.6) in $\Delta(z)$. So one has corresponding triples

$$\begin{pmatrix} \mathrm{Li}(y) \\ \log y \\ 1 \end{pmatrix}, \quad \begin{pmatrix} \hat{C}_2 \\ \hat{C}_1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} I(M_3) \\ \lambda(\gamma_1) \\ 1 \end{pmatrix}, \qquad \left(\text{and } \begin{pmatrix} \mathcal{L} \\ \mathrm{Re}\log \\ 1 \end{pmatrix}, \quad \begin{pmatrix} \mathrm{vol}(\Delta) \\ \mathrm{length}(\gamma) \\ 1 \end{pmatrix}\right). \tag{6.21}$$

[62] For M closed $CS(M)$ is essentially the η invariant, this is suitably extended for M non-compact.

Interpretation. Now having generalised our local G_2 holonomy manifold X_7 to a global manifold (compact and $K3$ fibered over $\mathbf{S}^3$) and furthermore having generalised the hyperbolic geometry of the simplex $\Delta(z)$ to smooth hyperbolic 3-manifolds let us indicate a potential connection. The quantum expression comprising all the corrections may have again a purely geometric description (when going to the dual description provided by the hyperbolic 3-manifold M, a 'thinned out' (spectral) version of the dual 7-fold just as the S/W curve is the $K3$-integrated-out version of the mirror CY). In the dual evaluation the membrane instanton superpotential $W = \mathrm{Li}(z)$, generalising $\mathcal{L}$, occurs as a complexified volume of a simplex in hyperbolic 3-space (respectively of a 3-manifold). So just as the periods of the S/W curve were periods of the type-IIB Calabi-Yau (mirror to the original theory in type IIA) now the dual 3-manifold $M^3_{\mathbf{u}}$ (analogue of the S/W curve E_u) and its 'period' $f(\mathbf{u}) = \int_{M^3_{\mathbf{u}}} \mathrm{vol} + i\mathcal{C}^{CS}$ (evaluated in the local case by $R(e^{\mathbf{u}})$, i.e. essentially by $\mathrm{Li}(e^{\mathbf{u}})$) reflects a ($W_G$?) 'period' (evaluated locally by $W \sim \mathrm{Li}(e^{\Phi})$).

Supersymmetric (associative) $\mathbf{S}^3$'s, which sit in X locally like in $\mathbf{S}^3 \times \mathbb{R}^4$, contribute to $H^3(X)$; in the dual 3-manifold M_3 the hyperbolic moduli space has dimension h, the number of ends, i.e. the number of link components in the description of the discriminant of a $K3$ fibration $X \to \mathbf{S}^3$ (responsible for the codimension 6 singularities); first these numbers and then the moduli of X_7 and M_3 have to be brought into relation for the dual description. Relating the moduli spaces[44] (cf. (6.1)) might include relations

$$\begin{aligned} \log u = \int_{\mathbf{S}^3_{Q_i}} -\Upsilon + iC &\longleftrightarrow \log\lambda_i = \int_{\gamma_i} \mathrm{length} + i\mathrm{tors}. \\ \frac{\partial W}{\partial \log u_k} = \log v_k &\longleftrightarrow \frac{\partial \mathcal{G}}{\partial u_i} = v_i \end{aligned} \tag{6.22}$$

(the coordinate $u = \exp\{i\int_Q C + i\Upsilon\}$ for the non-contractible $\mathbf{S}^3$ (and $\Phi_k = \log u$) are replaced with $\lambda = \exp\{\mathrm{length}(\gamma) + i\,\mathrm{tors}(\gamma)\}$ for the non-contractible $\mathbf{S}^1$ (and $\mathbf{u} = \log\lambda$)).

The base part $\mathbf{S}^3 - l$, over which the fibre is non-degenerate, is a 3-manifold M_3. Concerning the moduli spaces we want to compare note that for the superpotential we are interested in the number h_X of these $\mathbf{S}^3$ (related to codimension 7 singularities $C(\mathbf{S}^3 \times \mathbf{S}^3)$) whereas in the description of the hyperbolic 3-manifold and the complexified Chern-Simons invariant we are interested in the number $h = h_M$ of ends of M_3 (or components of the discriminant link describing codimension 6 singularities, related to $\mathbf{S}^4$ or the deformed $C(\mathbf{P}^3_{\mathbb{C}})$). The 3-manifold M_3 having three representations (the quotient $\Gamma\backslash\mathbf{H}_3$, a triangulation $M_3 = \cup_k^n \Delta(z_k)$ and the link complement $\mathbf{S}^3 - l$ of $l = \cup_j^h \gamma_j$) one now connects its second and third representation: the question of translating the different dimensions of moduli spaces is then captured by the reshuffling of the different summation boundaries in $K = \sum_i^h (\mathrm{length} + i\,\mathrm{tors})(\gamma_i)$ and $f = \sum_k^n R(z_k)$ (where R is essentially Li and for X_7, very naively, $W \approx \sum_j^{h_X} \mathrm{Li}(u_j)$) inherent in (6.16)–(6.19) [41].

The comparison (considered in more detail elsewhere [41]) will describe the actual form which the analogy between a local description of a singularity of a G_2 manifold by $X_7 = \mathbb{R}^4 \times \mathbf{S}^3$ and the Dehn filling of an end of a hyperbolic 3-manifold with the solid torus $\mathcal{T} = \mathbf{D^2} \times \mathbf{S}^1$ takes, i.e. the mapping between the moduli spaces of X_7 and M_3

(including the prepotential of hyperbolic deformation space) and the conncection between the membrane instanton superpotential and (possibly G-flux superpotentials resp.) the complex CS theory on M_3.

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A. The triality symmetry group on the moduli space

Σ_3, the permutation group of three elements, is built up from $\mathbb{Z}_2$ and an invariant subgroup $\mathbb{Z}_3 = \{e, \beta, \beta^2\}$; one has the relation $\alpha\beta^i\alpha = \beta^{-i}$ (read i mod 3).

$$1 \longrightarrow \mathbb{Z}_3 \longrightarrow \Sigma_3 \longrightarrow \mathbb{Z}_2 \longrightarrow 1\,. \tag{A.1}$$

The non-trivial coset consists of the three order two elements $\alpha, \alpha\beta, \alpha\beta^2$

$$\begin{array}{ccc} e & \beta & \beta^2 \\ \alpha & \alpha\beta & \alpha\beta^2\,. \end{array} \tag{A.2}$$

There are three conjugacy classes (CC) given by the elements of order one, two and three, respectively. We denote a conjugacy class by c, and the number of its elements by n_c. We index the classes by the common order of its elements, so $n_{c_1} = 1$, $n_{c_2} = 3$, $n_{c_3} = 2$.

Some representations are: the trivial representation $\mathbf{1} = triv$; the sign character

$$\text{sign}: \quad \Sigma_3 \to \Sigma_3/\mathbb{Z}_3 = \mathbb{Z}_2 = \{\pm 1\} \tag{A.3}$$

and the fundamental representation $\mathbf{3} = fund$ induced by permutation of $\{1, 2, 3\}$. In general, for a representation R, one has the following projection operators: first the 'invariant projector' $P^+(v) = \sum_{\gamma\in\Sigma_3} \gamma v$ (v in the representation space of R) which gives an Σ_3 invariant element; and analogously the 'anti-invariant projection'

$$P^-(v) = \sum_{\gamma\in\Sigma_3} \text{sign}(\gamma)\gamma v \tag{A.4}$$

which transforms with the sign character (both may be normalized by 1/6).

Note that the representation $\mathbf{3}$ is not irreducible. Think of it in real three-space to see the invariant (Euler) axis $\sum_i e_i$ and the $2\pi/3$ rotation in the orthogonal ('barycentric') plane. So it decomposes into a sum of the trivial representation and a two-dimensional irreducible representation, called $\mathbf{2}$ (we will also denote $-\mathbf{1} = $ sign and $-\mathbf{k} = -\mathbf{1} \otimes \mathbf{k}$)

$$\mathbf{3} = \mathbf{2} \oplus \mathbf{1}\,. \tag{A.5}$$

Let us denote the degree and character of a representation $\mathbf{d}$ by $\deg_{\mathbf{d}}$ and $\chi_{\mathbf{d}}$, respectively. The representations $\mathbf{1}, -\mathbf{1}, \mathbf{2}$ exhaust the irreducible representations as $3 = \sharp$ CC or

$$|\Sigma_3| = \deg^2_{\mathbf{1}} + \deg^2_{-\mathbf{1}} + \deg^2_{\mathbf{2}}\,. \tag{A.6}$$

At this point it might be appropriate to give the character table

	$\chi(c_1)$	$\chi(c_2)$	$\chi(c_3)$
$\mathbf{1}$	1	1	1
$-\mathbf{1}$	1	-1	1
$\mathbf{2}$	2	0	-1

Let us furthermore point to the following facts which we will use later

$$\mathbf{2}\otimes\mathbf{2} \cong \mathbf{2}\oplus\mathbf{1}\oplus-\mathbf{1} \tag{A.7}$$

$$\Lambda^2\mathbf{3} \cong \mathbf{2}\oplus-\mathbf{1} \tag{A.8}$$

$$\mathrm{Sym}^2\,\mathbf{3} \cong \mathbf{2}\oplus\mathbf{2}\oplus\mathbf{1}\oplus\mathbf{1}\,. \tag{A.9}$$

For (A.7) note[63] that the multiplicities $m_{\mathbf{d}}=1$ of each of our three building blocks $\mathbf{1},-\mathbf{1},\mathbf{2}$ occuring as isotypic components $\mathbf{d}$ in $\mathbf{2}\otimes\mathbf{2}$ follow from the character relations

$$m_{\mathbf{d}} = \frac{1}{|\Sigma_3|}\sum_{\gamma\in\Sigma_3}\chi_{\mathbf{2}\otimes\mathbf{2}}(\gamma)\overline{\chi_{\mathbf{d}}}(\gamma) = \frac{1}{|\Sigma_3|}\sum_{c\in CC} n_c\chi_{\mathbf{2}\otimes\mathbf{2}}(c)\overline{\chi_{\mathbf{d}}}(c) \tag{A.10}$$

Eq. (A.8) follows by inspection[64] and (A.9) from[65] $\mathrm{Sym}^2\,\mathbf{3}\cong(\mathbf{3}\otimes\mathbf{3})/\Lambda^2\mathbf{3}$.

Note also that if a system $(z_i)_{i\in\mathbb{Z}_3}$ spans a $-\mathbf{3}$, i.e. $\alpha z_i = -z_{\alpha i}$, then the system of $w_i := z_{i+1}-z_{i-1} = \beta^2 z_i - \beta z_i$ spans a $\mathbf{2}$ (by $\sum_i w_i = 0$ and $\alpha w_i = w_{\alpha i}$ from $\alpha\beta^2=\beta\alpha$)

$$\bigoplus_i z_i\mathbb{C}\cong-\mathbf{3}\Longrightarrow\sum_i w_i\mathbb{C}\cong\mathbf{2} \tag{A.11}$$

The f_i and α_i transform essentially (shifted by $\alpha\to\alpha\beta^2$) under $\mathbf{3}$ and $-\,\mathbf{3}$; then (A.11) leads to the introduction of the (log) y_i (similar the relation of the (log) η_i to the (log) y_i).

Some representation theory for Σ_3 acting on $\mathbf{P}^1$. For a Σ_3 action on $\mathbf{P}^1_{\mathbb{C}}$ consider the induced operation on functions[66] $\Lambda^0\mathbf{P}^1$ on $\mathbf{P}^1$. Consider now a $\mathbb{Z}_3$-orbit of an $\mathbb{Z}_2$-anti-invariant function f, i.e. of a function with $f(\alpha z) = -f(z)$, like the logarithm (here $\alpha z = 1/z$ like for the Sl_2 action). One has $\oplus_{i\in\mathbb{Z}_3}f(\beta_i\cdot)\mathbb{C}\subset\Lambda^0\mathbf{P}^1(-\mathbf{3})$, so

$$\bigoplus_i \log\beta^i z\,\mathbb{C}\cong-\mathbf{3}\,. \tag{A.12}$$

Now (3.2) implies for some $z:=\eta_i$

$$P^+_{\mathbb{Z}_3}\log z = \sum_{i\in\mathbb{Z}_3}\log\beta^i z\;=\;\pm\pi i \tag{A.13}$$

$$P^-\log z = \sum_{\gamma\in\Sigma_3}\mathrm{sign}(\gamma)\log\gamma z\;=\;\pm 2\pi i \tag{A.14}$$

[63] Or: $\mathbf{2}\otimes\mathbf{2}$ is represented by the span of $e_i\otimes e_j$ with $i=1,2; j=1,2$; clearly the diagonal provides a $\mathbf{2}$; the $+\mathbf{1}$ and $-\mathbf{1}$ are spaned by $e_1\otimes e_1+\frac{e_1\otimes e_2+e_2\otimes e_1}{2}+e_2\otimes e_2$ and $e_1\otimes e_2-e_2\otimes e_1$, respectively.

[64] For $\Lambda^2\mathbf{3}=\oplus_{i\in\mathbb{Z}_3}e_i\wedge e_{i+1}\mathbb{C}$ the split (A.5) leads now to the *anti*-invariant line $(\sum_{i\in\mathbb{Z}_3}e_i\wedge e_{i+1})\mathbb{C}$.

[65] Or: among the $e_i\cdot e_j$ $(i\le j)$ of $\mathrm{Sym}^2\,\mathbf{3}$ the diagonal and the $i<j$ part span each a $\mathbf{3}$.

[66] Functions are considered for now just formally, regardless of poles or the question of single-valuedness.

$$\sum_{i\in\mathbb{Z}_3} \log(-1)^{\delta_{ij}}\beta^i z = 0 \qquad (j\in\mathbb{Z}_3) \tag{A.15}$$

$$\sum_{i\in\mathbb{Z}_3} d\log\beta^i z = 0 \tag{A.16}$$

$$\sum_{i\in\mathbb{Z}_3} \mathrm{Re}\log\beta^i z = 0\,. \tag{A.17}$$

One has the exact sequence (by (A.13) the left term are the constants (= ker d) in the middle term)

$$\begin{array}{ccccccccc} 0 \longrightarrow & (\sum_i \log\beta^i z)\mathbb{C} & \longrightarrow & \oplus_i \log\beta^i z\,\mathbb{C} & \xrightarrow{d} & \sum_i (d\log\beta^i z\,\mathbb{C}) & \longrightarrow 0 \\ & \| & & \| & & \| & \\ 0 \longrightarrow & -\mathbf{1} & \longrightarrow & -\mathbf{3} & \longrightarrow & -\mathbf{2} & \longrightarrow 0 \end{array} \tag{A.18}$$

Similarly one has an interpretation of the *real* vector spaces with Σ_3 action (note (A.17))

$$\bigoplus_i \mathrm{Im}\log\beta^i z\mathbb{R} \cong -\mathbf{3} \tag{A.19}$$

$$\sum_i(\mathrm{Re}\log\beta^i z\mathbb{R}) \cong -\mathbf{2}\,. \tag{A.20}$$

A.1 Anti-invariance of $\mathcal{L}$: first argumen

We show how $d\,\mathrm{Li} = \log\beta z\, d\log z$ and $\mathrm{Im}\log\beta z\ \mathrm{Re}\log z$ behave under $e-\mathrm{sign}(\gamma)\gamma$. The complete parallelism shows (cf. footn. 68) that $(\mathrm{Im}\int d\,\mathrm{Li}) - \mathrm{Im}\log\beta z\ \mathrm{Re}\log z$ vanishes for all $e-\mathrm{sign}(\gamma)\gamma$ transformations, i.e. the anti-invariant transformation behaviour.

$\underline{\beta \cong \begin{pmatrix} \log z & \log\beta z & \log\beta^2 z \\ \log\beta z & \log\beta^2 z & \log z\end{pmatrix}}$: (the last equalities from (A.13))

$$\begin{aligned}
(e-\beta)d\,\mathrm{Li} &= \log\beta z d\log z - \log\beta^2 z d\log\beta z \\
&= d(\log\beta z\cdot\log z) - \log z d\log\beta z - \log\beta^2 z d\log\beta z \\
&= d(\log\beta z\cdot\log z) + \frac{1}{2} d\log^2\beta z \mp \pi i d\log\beta z \\
(e-\beta)\,\mathrm{Im}\log\beta z\,\mathrm{Re}\log z &= \mathrm{Im}\log\beta z\,\mathrm{Re}\log z - \mathrm{Im}\log\beta^2 z\,\mathrm{Re}\log\beta z \\
&= \mathrm{Im}(\log\beta z\cdot\log z) - \mathrm{Im}\log z\,\mathrm{Re}\log\beta z - \mathrm{Im}\log\beta^2 z\,\mathrm{Re}\log\beta z \\
&= \mathrm{Im}(\log\beta z\cdot\log z) + \frac{1}{2}\mathrm{Im}\log^2\beta z \mp \pi\,\mathrm{Im}\log\beta z\,.
\end{aligned}$$

$\underline{\beta^2 \cong \begin{pmatrix} \log z & \log\beta z & \log\beta^2 z \\ \log\beta^2 z & \log z & \log\beta z\end{pmatrix}}$: (the last equalities from (A.16))

$$\begin{aligned}
(e-\beta^2)d\,\mathrm{Li} &= \log\beta z d\log z - \log z d\log\beta^2 z \\
&= d(\log\beta z\cdot\log z) - \log z d\log\beta z - \log z d\log\beta^2 z \\
&= d(\log\beta z\cdot\log z) + \frac{1}{2} d\log^2 z \\
(e-\beta^2)\,\mathrm{Im}\log\beta z\,\mathrm{Re}\log z &= \mathrm{Im}\log\beta z\,\mathrm{Re}\log z - \mathrm{Im}\log z\,\mathrm{Re}\log\beta^2 z \\
&= \mathrm{Im}(\log\beta z\cdot\log z) - \mathrm{Im}\log z\,\mathrm{Re}\log\beta z - \mathrm{Im}\log z\,\mathrm{Re}\log\beta^2 z \\
&= \mathrm{Im}(\log\beta z\cdot\log z) + \frac{1}{2}\mathrm{Im}\log^2 z\,.
\end{aligned}$$

$\underline{\alpha \cong \begin{pmatrix} \log z & \log \beta z & \log \beta^2 z \\ -\log z & -\log \beta^2 z & -\log \beta z \end{pmatrix}}$: (the last equalities from (A.13))

$$\begin{aligned}
(e+\alpha)d\,\mathrm{Li} &= \log \beta z d \log z + \log \beta^2 z d \log z \\
&= -\frac{1}{2} d \log^2 z \pm \pi i d \log z \\
(e+\alpha)\,\mathrm{Im}\log \beta z\,\mathrm{Re}\log z &= \mathrm{Im}\log \beta z\,\mathrm{Re}\log z + \mathrm{Im}\log \beta^2 z\,\mathrm{Re}\log z \\
&= -\frac{1}{2}\,\mathrm{Im}\log^2 z \pm \pi\,\mathrm{Re}\log z\,.
\end{aligned}$$

$\underline{\alpha\beta \cong \begin{pmatrix} \log z & \log \beta z & \log \beta^2 z \\ -\log \beta z & -\log z & -\log \beta^2 z \end{pmatrix}}$:

$$\begin{aligned}
(e+\alpha\beta)d\,\mathrm{Li} &= \log \beta z d \log z + \log z d \log \beta z \\
&= d(\log \beta z \cdot \log z) \\
(e+\alpha\beta)\,\mathrm{Im}\log \beta z\,\mathrm{Re}\log z &= \mathrm{Im}\log \beta z\,\mathrm{Re}\log z + \mathrm{Im}\log z\,\mathrm{Re}\log \beta z \\
&= \mathrm{Im}((\log \beta z \cdot \log z)\,.
\end{aligned}$$

$\underline{\alpha\beta^2 \cong \begin{pmatrix} \log z & \log \beta z & \log \beta^2 z \\ -\log \beta^2 z & -\log \beta z & -\log z \end{pmatrix}}$: (the last equalities from (A.16) and (A.17))

$$\begin{aligned}
(e+\alpha\beta^2)d\,\mathrm{Li} &= \log \beta z d \log z + \log \beta z d \log \beta^2 z \\
&= -\frac{1}{2} d(\log^2 \beta z) \\
(e+\alpha\beta^2)\,\mathrm{Im}\log \beta z\,\mathrm{Re}\log z &= \mathrm{Im}\log \beta z\,\mathrm{Re}\log z + \mathrm{Im}\log \beta z\,\mathrm{Re}\log \beta^2 z \\
&= -\frac{1}{2}\,\mathrm{Im}(\log^2 \beta z)\,.
\end{aligned}$$

A.2 Anti-invariance of $\mathcal{L}$: second argument

To investigate the potential anti-invariant transformation behaviour of $\mathcal{L}(z)$ let us take up now our representation-theoretic considerations from section 3.

The proper reason for the anti-invariance of $\mathcal{L} = \mathrm{Im}\,\mathrm{Li}(z) - \mathrm{Im}\log \beta z\,\mathrm{Re}\log z$ is the following fact: when one operates on bilinear product expressions like $\log \beta^i z \log \beta^j z$ with either d or Im one finds as image elements in their respective target spaces (of expressions $\log \beta^i z d \log \beta^j z$ and $\mathrm{Im}\log \beta^i z\,\mathrm{Re}\log \beta^j z$) just the symmetric combinations by reason of the (pseudo-)derivative nature of these operations

$$\begin{aligned}
d(f \cdot g) &= df \cdot g + f \cdot dg \\
\mathrm{Im}(fg) &= \mathrm{Im}\, f\,\mathrm{Re}\, g + \mathrm{Re}\, f\,\mathrm{Im}\, g\,.
\end{aligned} \tag{A.21}$$

This is for $\mathrm{Li} = \int \log \beta z d \log z$ an indication that the integral can not be done elementary (the integrand is not symmetric, thereby not naturally a derivative of the presumptive candidate functions, cf. footn. 71). Now both terms, whose imaginary parts add up to $\mathcal{L}(z)$, i.e. $\int \log \beta z d \log z$ and $\psi = \log \beta z\,\mathrm{Re}\log z$ (or equally well $i\,\mathrm{Im}\log \beta z \log z$), constitute the one missing piece which, when linearly combined with the elementary expressions $\log \beta^i z \log \beta^j z$, gives after application of d and Im respectively not just the symmetric elements ($\mathrm{im}\, d$ and $\mathrm{im}\,\mathrm{Im}$ in (A.23) and (A.26), respectively) of their natural target space but all elements; furthermore both of these missing 'non-symmetric' elements are built from the same underlying element[67] $\log \beta z \otimes \log z$

$$
\begin{aligned}
d\,\mathrm{Li} &= \log \beta z d \log z \\
\mathrm{Im}\,\psi &= \mathrm{Im} \log \beta z \,\mathrm{Re} \log z
\end{aligned}
\tag{A.22}
$$

We will see in a moment that the non-vanishing elements in the respective one-dimensional quotient space (target space modulo image) they generate transform with the sign character (A.24), (A.27). The common origin of the non-trivial terms and the complete parallelism of the (pseudo-)derivative operations mentioned above shows then that the classes, when lifted back to the proper elements, acquire exactly *the same* correction terms which gives finally the anti-invariance of their difference.[68]

Now consider the following exact diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \downarrow & & & & & & \\
 & & \mathbb{C} & & & & & & \\
 & & \downarrow & & & & & & \\
0 & \longrightarrow & \mathrm{Sym}^2\,\mathrm{fund} & \longrightarrow & \mathrm{Sym}^2\,\mathrm{fund} \oplus \mathrm{Li}\,\mathbb{C} & \longrightarrow & \mathrm{Li}\,\mathbb{C} & \longrightarrow & 0 \\
 & & \downarrow d & & \downarrow d & & \downarrow d & & \\
0 & \longrightarrow & \mathrm{im}\,d & \longrightarrow & \oplus_{i,j} \log \beta^i z d \log \beta^j z \mathbb{C} & \longrightarrow & [d\,\mathrm{Li}]\mathbb{C} & \longrightarrow & 0 \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}
\tag{A.23}
$$

From consideration of the lower horizontal exact sequence one finds

$$
[d\,\mathrm{Li}]\mathbb{C} \cong -\mathbf{1}\,. \tag{A.24}
$$

For, by (A.18), the space of elementary bilinear expressions $\log \beta^i z \log \beta^j z$ gives a $\mathrm{Sym}^2(-\mathbf{3}) = \mathrm{Sym}^2\,\mathbf{3}$; concerning $\mathrm{im}\,d$ note that the kernel of constants in the vertical short exact sequence is, by (A.13), $(\sum_i \log \beta^i z)^2\mathbb{C} \cong \mathrm{Sym}^2(-\mathbf{1}) = \mathbf{1}$; so by (A.9)

$$
\mathrm{im}\,d \cong \mathbf{2} \oplus \mathbf{1} \oplus \mathbf{2}\,. \tag{A.25}
$$

The middle term in the lower sequence, is given by[69] $-\mathbf{3} \otimes -\mathbf{2} = \mathbf{3} \otimes \mathbf{2} = \mathbf{2} \otimes \mathbf{2} \oplus \mathbf{2}$. Thereby $[d\,\mathrm{Li}]\mathbb{C} \cong (-\mathbf{3} \otimes -\mathbf{2}) / \mathrm{im}\,d = (\mathbf{2} \otimes \mathbf{2} \oplus \mathbf{2}) / (\mathbf{2} \oplus \mathbf{1} \oplus \mathbf{2})$; (A.7) now implies (A.24).

[67] Of the tensor product which one has to take, instead of the symmetric product used above in the elementary expressions, to be able to apply d or Im to individual factors.

[68] When going back and forth in $\mathrm{Im} \circ d^{-1}$ the interrelations are kept, i.e. the integration constants are real (actually rational multiples of $(\pi i)^2$) (note that $ker d = \mathbb{C}$, $ker Im = \mathbb{R}$ on holomorphic functions).

[69] The tensor (instead of the symmetric) product applies as the symmetry between the factors is broken.

Now consider the following companion exact diagram

$$\begin{array}{ccccccccc} & & 0 & & & & & & \\ & & \downarrow & & & & & & \\ & & \mathbb{C} & & & & & & \\ & & \downarrow & & & & & & \\ 0 & \longrightarrow & \mathrm{Sym}^2\,\mathrm{fund} & \longrightarrow & \mathrm{Sym}^2\,\mathrm{fund} \oplus \psi\mathbb{C} & \longrightarrow & \psi\mathbb{C} & \longrightarrow & 0 \\ & & \downarrow \mathrm{Im} & & \downarrow \mathrm{Im} & & \downarrow \mathrm{Im} & & \\ 0 & \longrightarrow & \mathrm{im}\,\mathrm{Im} & \longrightarrow & \oplus_{i,j} \,\mathrm{Im} \log \beta^i z \,\mathrm{Re} \log \beta^j z \mathbb{R} & \longrightarrow & [\mathrm{Im}\,\psi]\mathbb{R} & \longrightarrow & 0 \\ & & \downarrow & & & & & & \\ & & 0 & & & & & & \end{array} \tag{A.26}$$

By completely parallel arguments, where now the second embodiment of $-\mathbf{2}$ in (A.20) replaces the first one (A.18) used before, here too one finds (as representations over $\mathbb{R}$)

$$[\mathrm{Im}\,\psi]\mathbb{R} \cong -\mathbf{1}\,. \tag{A.27}$$

A.3 Formal anti-invariance of a modified superpotential

To understand the anti-invariance property of $\tilde{W}$ in (4.19) one would like to see the corresponding symmetry becoming manifest. This can be done on the derivative level: dW/dz becomes an elementary logarithmic function just as the correction terms dC/dz, with the only difference that C (in contrast to W) is already itself an elementary function. To avoid an additional transformation factor $d(\gamma z)/dz$ obscuring the transformation properties, we actually consider the one-form $d\tilde{W}$ which again transforms with the sign character: i.e. we are using the Σ_3 equivariant map[70] $d : \Lambda^0 \mathbf{C_z} \longrightarrow \Lambda^1 \mathbf{C_z}$.

One finds (cf. (A.32)) for $d\tilde{W}$ the manifestly anti-invariant expression

$$6d\tilde{W} = \sum_{i\in\mathbb{Z}_3} \log \frac{\beta^{i+1}z}{\beta^{i-1}z} d\log\beta^i z \qquad \left(= \sum_{i\in\mathbb{Z}_3} d\log \frac{\beta^{i-1}z}{\beta^{i+1}z} \,\log\beta^i z \right). \tag{A.28}$$

The first of the six terms is just the original term we started with

$$dW = \log\beta u \; d\log u\,. \tag{A.29}$$

From (A.29) one can read off directly that W is not anti-invariant (compare to (A.28). Note that our solution (A.28) is actually indeed of the form (4.20) (by integration), i.e.

$$6d\tilde{W}(\cdot) = \sum_{\gamma\in\Sigma_3} \mathrm{sign}(\gamma)(dW)(\gamma\cdot) = \sum_{\gamma\in\Sigma_3} \mathrm{sign}(\gamma) d(W(\gamma\cdot))\,. \tag{A.30}$$

Matching to (A.28) is obvious for $\gamma \in \mathbb{Z}_3$ by (A.29) which also gives $(dW)(\alpha z) = \log\beta^2 z \times d\log z$. (A non-zero integration constant would violate $\tilde{W}(\gamma\cdot) = \mathrm{sign}(\gamma)\tilde{W}(\cdot)$.)

[70] Meaning $(df)(\gamma\cdot) = (\frac{df}{dz}dz)(\gamma\cdot) = \frac{df}{dz}(\gamma\cdot)d(\gamma\cdot) = d(f(\gamma\cdot))$.

Now, to prove (A.28), one finds from (4.21)

$$
\begin{aligned}
6dC = & -\log z\, d\log z \quad - 2\log \beta z\, d\log z \\
& -2\log z\, d\log \beta z + 3\log\beta z\, d\log\beta z\; + 2\log\beta^2 z\, d\log\beta z \\
& \qquad +2\log\beta z\, d\log\beta^2 z + \log\beta^2 z\, d\log\beta^2 z\,.
\end{aligned}
\tag{A.31}
$$

Combining with (A.29) one gets[71] (A.28) after a regrouping in the 'verticals' via (A.16)

$$
\begin{aligned}
6d\tilde{W} = & \qquad\qquad + \log\beta z\, d\log z \; - \log\beta^2 z\;\; d\log z\, - \\
& - \log z\, d\log\beta z \qquad\qquad\qquad + \log\beta^2 z\, d\log\beta z\, + \\
& + \log z\, d\log\beta^2 z - \log\beta z\, d\log\beta^2 z\,.
\end{aligned}
\tag{A.32}
$$

B. The monodromy representation

The polylogarithms are (with $\mathrm{Li}(x) := \mathrm{Li}_2(x)$, $\mathrm{Li}_1(x) = \log\beta x$, $\mathrm{Li}_0(x) = x\cdot\beta x$)

$$
\mathrm{Li}_k(x) = \sum_{k\geq 1}\frac{x^n}{n^k}\,, \qquad \frac{d}{dx}\mathrm{Li}_{k+1}(e^x) = \mathrm{Li}_k(e^x)\,. \tag{B.1}
$$

To express the multi-valuedness of $W = \mathrm{Li}_2$ define the matrix differential form [18]

$$
\Omega = \begin{pmatrix} 0 & d\log\beta z & 0 \\ 0 & 0 & d\log z \\ 0 & 0 & 0 \end{pmatrix}. \tag{B.2}
$$

The one-forms $\omega_i = d\log\beta^i z$ are related with the loops l_i by $\frac{1}{2\pi i}\int_{l_j}\omega_k = \delta_{jk}$. Now consider for a (multi-valued) function $F : \mathbf{P}^1\backslash\{0,1,\infty\} \to gl(3,\mathbb{C})$ the matrix differential equation

$$
dF = F\cdot\Omega\,. \tag{B.3}
$$

A fundamental solution is provided by the principal branch (on $|z - 1/2| < 1/2$) of

$$
L(z) = \begin{pmatrix} 1 & \log\beta z & \mathrm{Li}(z) \\ 0 & 1 & \log z \\ 0 & 0 & 1 \end{pmatrix}. \tag{B.4}
$$

Analytic continuation of the principal branch of $L(z)$ about a loop l in $\mathbf{P}^1\backslash\{0,1,\infty\}$ (based at $1/2$, say) leads to another fundamental solution $M(l)L(z)$ where

$$
M : \pi_1(\mathbf{P}^1\backslash\{0,1,\infty\}) \to \mathrm{Gl}(3,\mathbb{C}) \tag{B.5}
$$

defines the monodromy representation. One finds for the images of the generator loops $l_i(t)$ $(i = 0,1)$ the representing matrices[72] $M(l_i)$ in (4.33).

[71] Note the anti-symmetry of the coefficient matrix which guarantees the non-triviality of the expression (of course, being non-symmetric is enough; if one starts form an elementary expression $F = \sum_{i,j} a_{i,j}\log\beta^i z\log\beta^j z$ one ends up with a symmetric quantity $dF = \sum_{i,j}(a_{i,j}+a_{j,i})\log\beta^i z d\log\beta^j z$. The missing symmetry did account already for the non-triviality of the original term (A.29); cf. (A.23)).

[72] Multiplying the rows $r^{(j)}$ by $(2\pi i)^{j-1}$ the factor $2\pi i$ can be put in (B.4). The $r^{(j)}$ are multi-valued but the $\mathbb{Q}$-linear span of the $(2\pi i)^{j-1}r^{(j)}$ is well-defined (the monodromy representation is then rational.).

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In a column *vector picture* the three columns c_k, $k = 1, 2, 3$, of L fulfill $d\, c_k = c_k \cdot \Omega$ and one gets from (4.33) the monodromies (4.31) for c_3. Similarly in the row picture the rows $r^{(j)}$ $(j = 1, 2, 3)$ in (B.4) are *flat* sections[72] of a meromorphic connection ∇ (on the trivial $\mathbf{C^3}$ bundle over $\mathbf{P}^1$) given for a section $s = (s_1, s_2, s_3) : \mathbf{P}^1 \backslash \{0, 1, \infty\} \to \mathbf{C^3}$ by

$$\nabla s = ds - s\Omega = (ds_1, ds_2 - s_1 d \log \beta z, ds_3 - s_2 d \log z)\,. \tag{B.6}$$

The *Heisenberg picture* involves the complexified Heisenberg group. Consider first the situation over the reals with the following central extension of the group $(\mathbb{R}^2, +)$ by $(\mathbf{S}^1, \cdot)$

$$1 \longrightarrow \mathbf{S}^1 \longrightarrow \mathcal{H} \longrightarrow \mathbb{R}^2 \longrightarrow 0\,. \tag{B.7}$$

So the normal subgroup $\mathbf{S}^1$ of $\mathcal{H}$ constitutes the centre and one has the group law

$$(X, \lambda) \cdot (Y, \mu) = (X + Y, e(X, Y)\lambda\mu) \tag{B.8}$$

with a skew-multiplicative[73] pairing[74] $e : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbf{S}^1$ given by $e(X, Y) = e^{2\pi i A(X,Y)}$ for $A : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ a non-degenerate, bilinear, skew-symmetric pairing. With the parameterisation $\lambda = e^{2\pi i c}, \mu = e^{2\pi i d}$ one finds as multiplication law on $\mathbb{R} \times \mathbb{R}^2$

$$(x_1, x_2 \,|\, c) \cdot (y_1, y_2 \,|\, d) = (x_1 + y_1, x_2 + y_2 \,|\, A(X, Y) + c + d) \tag{B.9}$$

Choosing for A the pairing $A(X, Y) = x_1 y_2 - x_2 y_1$ (for $X = (x_1, x_2), Y = (y_1, y_2)$) one sees that the group law (B.9) on triples $(x_1, x_2 \,|\, c) \in \mathbb{R}^2 \times \mathbb{R} := \mathcal{H}'$ is induced from matrix multiplication under the following association of $\mathcal{H}'$ with the upper triangular matrices

$$(a, b \,|\, c) \cong \begin{pmatrix} 1 & a & \frac{c+ab}{2} \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \Longrightarrow (a, b \,|\, c) \cdot (u, v \,|\, w) = (a + u, b + v \,|\, av - bu + c + w)$$

Note that one has a slightly different induced group law by the following association

$$(a, b \,|\, c) \cong \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \Longrightarrow (a, b|c) \cdot (u, v|w) = (a + u, b + v|av + c + w)\,. \tag{B.10}$$

Define the *complexified* Heisenberg group $\mathcal{H}_\mathbf{C}$ (with the $\mathbf{S}^1$ from $e^{2\pi i(\cdot)}$ replaced by $\mathbb{C}^*$) where $\mathcal{H}_\mathbf{C}$ is $\mathbf{C^3}$ with this composition (and so with inverse $(a, b \,|\, c)^{-1} = (-a, -b \,|\, ab - c)$) which makes $\mathcal{H}_\mathbf{C}$ a non-commutative group with normal subgroups $(*, 0 \,|\, *')$ and $(0, * \,|\, *')$, both isomorphic to $(\mathbf{C^2}, +)$, whose intersection $(0, 0 \,|\, *)$ is the centre of $\mathcal{H}_\mathbf{C}$.

The adjusted imaginary part $\mathcal{L}$ of W Consider the Heisenberg bundle (4.39) with section s (where $e(\mathcal{H}_\mathbf{Z}(a, b \,|\, c)) = (e^a, e^b)$)

$$\begin{array}{ccc} & & (2\pi i)^2 \mathbb{Z} \backslash \mathbf{C_c} \\ & & \downarrow \\ \underline{\mathcal{H}} & \longrightarrow & \mathcal{H}_\mathbf{Z} \backslash \mathcal{H}_\mathbf{C} \\ s \uparrow\downarrow pr & & \downarrow e \\ \mathbf{P}^1 \backslash \{0, 1, \infty\} & \overset{(1-z,z)}{\longrightarrow} & \mathbb{C}^* \times \mathbb{C}^* \end{array} \tag{B.11}$$

$$s(z) = \mathcal{H}_\mathbf{Z}(-\log \beta z, \log z \,|\, c)\,. \tag{B.12}$$

[73] So one has $e(X + X', Y) = e(X, Y)e(X', Y)$, similarly in Y and $e(Y, X) = e(X, Y)^{-1}, e(X, X) = 1$.

[74] $X \to e(X, \cdot)$ will then provide an isomorphism of $\mathbb{R}^2$ with its character group.

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The $\mathcal{H}_{\mathbf{Z}}$ coset expresses the fact that $z \to c$ is not given as a function (the monodromy increments of Li). This comes as the right vertical sequence in (B.11) does not split, i.e. there is no map α with $(0,0\,|\,c) \to \mathcal{H}_{\mathbf{Z}}(0,0\,|\,c) \xrightarrow{\alpha} (0,0\,|\,c)$. For the imaginary part of the fibre $(2\pi i)^2\mathbb{Z}\backslash\mathbf{C_c} = \Big((2\pi)^2\mathbb{Z}\backslash\mathbb{R}_{\mathrm{Re}\,c}\Big) \oplus i\mathbb{R}_{\mathrm{Im}\,c}$ there is such a map. The function f on $\mathcal{H}_{\mathbf{C}}$

$$f(u,v,w) = \mathrm{Im}\,w - \mathrm{Re}\,u\,\mathrm{Im}\,v \tag{B.13}$$

is invariant under action of $\mathcal{H}_{\mathbf{R}}$ from the left, so a fortiori under $\mathcal{H}_{\mathbf{Z}}$ which according to the remark after (4.35) represents the monodromy increments. So [19] the combination

$$\mathcal{L} : \mathbf{P}^1\backslash\{0,1,\infty\} \ni z \xrightarrow{\Lambda} \mathcal{H}_{\mathbf{Z}}\left(-\frac{\log\beta z}{2\pi i}, \frac{\log z}{2\pi i}, -\frac{\mathrm{Li}(z)}{(2\pi i)^2}\right) \in \mathcal{H}_{\mathbf{Z}}\backslash\mathcal{H}_{\mathbf{C}} \xrightarrow{-(2\pi i)^2 f} \mathbb{R} \tag{B.14}$$

(making factors $2\pi i$ manifest) gives a π_1-invariant, i.e. single-valued function (4.40).

Some expressions related to Li **and the Rogers CFT relation.** The $\mathbb{Z}_2$ anti-projector $P^-_{\mathbb{Z}_2}f = \frac{1}{2}\sum_{i\in\mathbb{Z}_2}\mathrm{sign}(\gamma^i)\gamma^i f = (f-\gamma f)/2$ does not reproduce Li (not anti-invariant; $\gamma \in \Sigma_3\backslash\mathbb{Z}_3$), so we introduce the Rogers function R (for $\gamma = \alpha\beta$)

$$R(z) = \frac{1}{2}\left(\mathrm{Li}(z) - \mathrm{Li}(1-z)\right) + \frac{\pi^2}{12} = \mathrm{Li}(z) - \frac{1}{2}\log\beta z \log z \tag{B.15}$$

$$\begin{aligned}
\mathrm{Li}(z) &= \int_0^z \log\beta w d\log w\,, & d\,\mathrm{Li}(z) &= \log\beta z d\log z\\
\mathcal{L}(z) &= \mathrm{Im}\,\mathrm{Li}(z) - \mathrm{Im}\log\beta z\,\mathrm{Re}\log z\,, & di\mathcal{L}(z) &= \frac{1}{2}(\mathrm{Re}\log\beta z d\log z - \mathrm{Re}\log z d\log\beta z)\\
R(z) &= \mathrm{Li}(z) - \frac{1}{2}\log\beta z\log z\,, & dR(z) &= \frac{1}{2}(\log\beta z d\log z - \log z d\log\beta z)\,.
\end{aligned} \tag{B.16}$$

For background on (5.11) recall that Calabi-Yau hypersurfaces in weighted projective space have a Gepner point in their moduli space with the underlying exactly solvable RCFT a tensor product of $N = 2$ superconformal minimal models of central charge[75] $c = \frac{3k}{k+2}$ (the central charge of an integrable level k representation of the affine Kac-Moody Lie algebra of Sl(2)). Recall the character $\chi_n(\theta) = \sin(n\pi\theta)/\sin(\pi\theta)$ of the n-dimensional representation of SU(2). The characters $\chi_i(\tau,z) = \mathrm{tr}_{\mathcal{H}_i}\, q^{L_0-\frac{c}{24}}q^{2\pi i u J_0}$ transform like $\chi_i(\frac{-1}{\tau},\frac{z}{\tau}) = e^{\pi i k z^2/2}\sum_j S_{ij}\chi_j(\tau,z)$ (Q_{ij} generalized quantum dimensions)

$$S_{ij} = \sqrt{\frac{2}{k+2}}\sin(i+1)(j+1)\frac{\pi}{k+2}\,, \qquad Q_{ij} = \frac{S_{ij}}{S_{0j}} = \frac{\sin(i+1)(j+1)\frac{\pi}{k+2}}{\sin(j+1)\frac{\pi}{k+2}}$$

giving (j fixed)

$$\sum_{i=1}^{k} R(1/Q_{ij}^2) = \frac{\pi^2}{6}\left(\frac{3k}{k+2} - 24\Delta_j^{(k)} + 6j\right).$$

$j = 0$, $N = k+2$ give (5.11) [42].[76]

[75] For $\sum_{i=0}^4 z_i^{a_i} = 0$ in $\mathbf{P}^4_{(w_i)}(d)$ with $a_i = \frac{d}{w_i}$ the CFT is a suitably interpreted tensor product of five SU(2) theories of level $a_i - 2$ and chiral primary operators with integral anomalous dimensions come from operators in the $\mathrm{SU}(2)_{k=a_i-2}$ factors with anomalous dimensions $\Delta_j^{(k)} = \frac{j(j+2)}{4(k+2)}$ ($j = 0,\ldots,k$).

[76] Concerning Q_{i0} recall that $Z(\mathbf{S}^2\times\mathbf{S}^1) = 1, Z(\mathbf{S}^3) = S_{0,0}$ give for the vev $\langle C\rangle = \frac{\sin N\frac{\pi}{N+k}}{\sin\frac{\pi}{N+k}}$ of the unknot as Wilson line in $\mathbf{S}^3$ (for $G = \mathrm{SU}(N)$) that $\langle C\rangle = \frac{Z(\mathbf{S}^3,R_2)}{Z(\mathbf{S}^3)} = \frac{S_{0,1}}{S_{0,0}}$ for $G = \mathrm{SU}(2)$.

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C. Volume of a hyperbolic ideal tetrahedron

Hyperbolic three-space. As model for the hyperbolic space $\mathbf{H}_3$ we take the half-space model constructed in analogy to the upper half-plane $\mathbf{H} = \{\mathbf{x} = x_1 + x_2 i \mid x_1, x_2 \in \mathbb{R}, x_2 > 0\}$; we consider $\mathbb{C}$ embedded at $x_3 = 0$ so that $\mathbf{H}_3$ is $\{(w := x_1 + ix_2, t := x_3) \in \mathbb{C} \times \mathbb{R}^{>0}\}$. Now, $\mathbf{H}$ is also a homogeneous space $\mathrm{PSl}(2,\mathbb{R})/\,\mathrm{SO}(2)$ from the operation of $\mathrm{Sl}(2,\mathbb{R})$ on i by fractional linear transformations. Consider here the following subspace of the quaternions

$$\mathbf{H}_3 = \{\mathbf{x} = x_1 + x_2 i + x_3 j \mid x_1, x_2, x_3 \in \mathbb{R}, x_3 > 0\} \tag{C.1}$$

$g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sl}(2,\mathbb{C})$ operates on $\mathbf{H}_3$ by

$$g \cdot \mathbf{x} = (a\mathbf{x} + b)(c\mathbf{x} + d)^{-1}\,. \tag{C.2}$$

With the norm $||c(w+tj)+d||^2 = |cw+d|^2 + |c|^2 t^2$ in the quaternions this is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(w,t) = \frac{1}{||c(w+tj)+d||^2}\Big((aw+b)(\bar{c}\bar{w}+\bar{d}) + |c|^2 t^2, t\Big)\,. \tag{C.3}$$

So for $z \in \mathbb{C}^\times \hookrightarrow \mathrm{Sl}(2,\mathbb{C})$ (via $z = a^2, b = c = 0$) one has $z \cdot (w,t) = (zw, |z|t)$.

Now just as for $\mathbf{H} = \mathbf{H}_2$ one has here that the map $q : \mathrm{Sl}(2,\mathbb{C}) \to \mathbf{H}_3$ given by $g \xrightarrow{q} g \cdot j$ induces an equivariant diffeomorphism $q : \mathrm{Sl}(2,\mathbb{C})/\,\mathrm{SU}(2) \xrightarrow{\cong} \mathbf{H}_3$

$$1 \longrightarrow \mathrm{SU}(2) \xrightarrow{r} \mathrm{Sl}(2,\mathbb{C}) \xrightarrow{q} \mathbf{H}_3 \longrightarrow 1\,. \tag{C.4}$$

This may be equally well expressed by considering the quotient $\mathrm{PSl}(2,\mathbb{C})/\,\mathrm{SO}(3)$.

Prolonging the analogy to the real case note that the boundary of the upper half plane is identified with $\mathbf{P}^1_{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ with $\mathbb{R}$ the locus $x_2 = 0$ whereas here the boundary is $\mathbf{P}^1_{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ with $\mathbb{C}$ the locus $x_3 = 0$. The group of (orientation preserving) isometries of $\mathbf{H}$ is isomorphic to $\mathrm{PSl}(2,\mathbb{R})$ and for $\mathbf{H}_3$ to[77] $\mathrm{PSl}(2,\mathbb{C})$.

The standard hyperbolic metric ds^2 and the volume form vol are given by

$$ds^2 = \frac{dx_1^2 + dx_2^2 + dx_3^2}{x_3^2}\,, \qquad \mathrm{vol} = \frac{dx_1 dx_2 dx_3}{x_3^3}\,. \tag{C.5}$$

For example, using the mentioned embedding $z \in \mathbb{C}^\times \hookrightarrow \mathrm{Sl}(2,\mathbb{C})$, one finds for the length of the geodesic line-segment γ_z from $q(e)$ to $q(z)$ (with $e = \mathbf{1_2} \in \mathrm{Sl}(2,\mathbb{C})$)

$$\mathrm{length}(\gamma_z) = \mathrm{distance}(j, |z|j) = \int_1^{|z|} \frac{dx_3}{x_3} = \mathrm{Re}\log z\,. \tag{C.6}$$

[77]The latter acts on the boundary $\mathbf{P}^1_{\mathbb{C}}$ fractionally linear, so acts three-fold transitively and maps (uniquely) two quadruples of points onto another exactly if they have the same cross ratio.

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Volume of a hyperbolic ideal tetrahedron. Now an ideal tetrahedron is determined (up to congruence) by the dihedral angles $\gamma_1, \gamma_2, \gamma_3$ of the edges incident to any vertex. Then choosen *any* vertex[78] one has

$$\sum_i \gamma_i = \pi\,. \tag{C.7}$$

We will choose the vertex at ∞ so that the angles become angles of an euclidean triangle in[79] $\mathbb{C}$ given by the remaining three vertices u, v, w.[80]

Now concerning the parametrization of an euclidean triangle $\Delta(u,v,w) \subset \mathbb{C}$ (the vertices labeled in the mathematical positive sense) note that if one associates to each vertex the ratio of the adjacent sides

$$z(u) = \frac{w-u}{v-u}\,, \qquad z(v) = \frac{u-v}{w-v} = \beta z(u)\,, \qquad z(w) = \frac{v-w}{u-w} = \beta^2 z(u) \tag{C.8}$$

then these *vertex invariants* depend only on the orientation preserving similarity class of $\Delta(u,v,w)$ which in turn is determined by $z(u)$ ($\arg z(u)$ is the angle of $\Delta(u,v,w)$ at u; $\operatorname{Im} z(u) > 0$). So after the usual normalization in our tetrahedron set-up we are considering the angles of the euclidean triangle with vertices $0, 1, z$ in $\mathbb{C}$, the angle α_0 at 0 is $\arg z$ and the angles α_1 at 1 and α_z at z are given by (cf. footn. 37)

$$\alpha_0 = \arg z\,, \qquad \alpha_1 = \arg \beta z\,, \qquad \alpha_z = \arg \beta^2 z\,. \tag{C.9}$$

In other words this gives a geometric manifestation of the membrane anomaly (3.2)

$$\sum_{i\in\mathbb{Z}_3} \operatorname{Im}\log \beta^i z = \pi \tag{C.10}$$

(cf. (1.9), (2.19)). As $z, \beta z, \beta^2 z$ give the same tetrahedron one must pick an edge of Δ (the dihedral angle of the faces adjacent at this edge is then $\arg z$) to specify z uniquely.

Furthermore one has with (4.25) that (where $\gamma_{1,2,3} = \alpha_{0,1,z}$)

$$\operatorname{vol}\,\Delta(z) = \sum_i \Pi(\gamma_i)\,. \tag{C.11}$$

For convenience let us choose a slightly different 'circle gauge' of the points z_i: the one actual complex degree of freedom (which is left after the $\mathrm{Sl}(2,\mathbb{C})$ operation) will not be encoded in the complex number z with the other three points fixed (leaving two real degrees of freedom); rather we gauge to a situation where one of the points again becomes ∞ and the other three points a, b, c lie on the unit circle $|z| = 1$ (these are three real degrees of freedom) with the one further condition that $\operatorname{Re} b = \operatorname{Re} c$ (leaving two real degrees of freedom). In this situation, where we assume that the face opposite to ∞ lies

[78]As opposite dihedral angles are equal the γ_i are independent of the vertex chosen.

[79]The corresponding face of Δ is a hemisphere *over* $\mathbb{C}$ through u, v, w bounded by semi-circles over $\mathbb{C}$.

[80]The *link* L (parametrizing the rays in Δ through v) of a vertex v of an ideal tetrahedron Δ is an euclidean triangle (well-defined up to orientation preserving similarity) given by the intersection of the boundary of Δ with a horizontal euclidean plane (a "horosphere"); L determines Δ up to congruence.

in the hemisphere $x_1^2+x_2^2+x_3^2=1$ ($x_3\geq 0$) and has vertices a,b,c (of $x_3=0$), project Δ orthogonally down to the unit disk D_{x_1,x_2} where you get the picture of an euclidean triangle (of vertices a,b,c) whose angles sum up to π. Subdividing this triangle by drawing the heights from the origin on the sides one gets six smaller right triangles. Then one computes with (C.5) for the volume vol_1 of the region lying over one of these triangles (with angle γ, $A^2:=1-x_1^2$, $\cos\theta=x_1$ and (4.26)) ([21])

$$\begin{aligned}\mathrm{vol}_1 &= \int_0^{\cos\gamma} dx_1 \int_0^{x_1\tan\gamma} dx_2 \int_{\sqrt{1-x_1^2-x_2^2}}^{\infty} \frac{dx_3}{x_3^3} = \frac{1}{2}\int_0^{\cos\gamma} dx_1 \int_0^{x_1\tan\gamma} \frac{dx_2}{1-x_1^2-x_2^2} \\ &= \frac{1}{4}\int_0^{\cos\gamma} dx_1 \frac{1}{A}\log\frac{A+x_1\tan\gamma}{A-x_1\tan\gamma} = -\frac{1}{4}\int_{\pi/2}^{\gamma} d\theta \log\frac{2\sin(\theta+\gamma)}{2\sin(\theta-\gamma)} \\ &= \frac{1}{4}\left(\Pi(2\gamma)+\Pi\left(\frac{\pi}{2}-\gamma\right)-\Pi\left(\frac{\pi}{2}-\gamma\right)-\Pi(0)\right) = \frac{1}{2}\Pi(\gamma)\,. \end{aligned} \tag{C.12}$$

By summing over the six partial triangles one gets thereby (C.11). This gives the connection of (4.43) with (4.45) in view of the remark following (C.7).

Cross ratios. The Σ_3 transformation properties of a cross ratio can be understood as follows. For four points z_1, z_2, z_3, z_4 of $\mathbf{P}^1(\mathbb{C})$ one defines their cross ratio

$$cr\{z_1,z_2,z_3,z_4\} = \frac{z_1-z_3}{z_1-z_4}\Big/\frac{z_2-z_3}{z_2-z_4} \tag{C.13}$$

For example $cr\{0,1,\infty,z\}=z$. Clearly a Σ_4 is operating. One has the equalities

$$cr\{z_1,z_2,z_3,z_4\} = cr\{z_2,z_1,z_4,z_3\} = cr\{z_3,z_4,z_1,z_2\} = cr\{z_4,z_3,z_2,z_1\} \tag{C.14}$$

but the index four subgroup Σ_3 operates effectively which gives the following realisation

$$\begin{aligned} x &= cr\{z_1,z_2,z_3,z_4\} & \frac{1}{1-x} &= cr\{z_1,z_3,z_4,z_2\} & \frac{x-1}{x} &= cr\{z_1,z_4,z_2,z_3\} \\ \frac{1}{x} &= cr\{z_1,z_2,z_4,z_3\} & 1-x &= cr\{z_1,z_3,z_2,z_4\} & \frac{x}{x-1} &= cr\{z_1,z_4,z_3,z_2\} \end{aligned} \tag{C.15}$$

of the isomorphism $\Sigma_3\cong \mathrm{Sl}(2,\mathbb{Z})/\Gamma(2)$ in

$$1\longrightarrow V\to\Sigma_4\longrightarrow\Sigma_3\cong\mathrm{Sl}(2,\mathbb{Z})/\Gamma(2)\longrightarrow 1\,. \tag{C.16}$$

D. Cohomological interpretation

Eq. (6.12) gives for the classifying space $\mathrm{BGl}(n,\mathbb{C})^\delta$ of flat bundles an universal class

$$\hat{C}_2\in H^3(\mathrm{BGl}(n,\mathbb{C})^\delta,\mathbb{C}/\mathbb{Z})\cong H^3_{EM}(\mathrm{Gl}(n,\mathbb{C})^{\mathrm{disc}},\mathbb{C}/\mathbb{Z}) \tag{D.1}$$

where we also indicated the isomorphism of the topological homology with the Eilenberg-MacLane group cohomology[81] of the underlying discrete group of $\mathrm{Gl}(n,\mathbb{C})$.

[81]Homology is of chain complex of elements of G^n with boundary $\partial(g_1,\ldots,g_n)=(g_2,\ldots,g_n)+\sum_{i=1}^{n-1}(-1)^i(g_1,\ldots,g_ig_{i+1},\ldots,g_n)+(-1)^n(g_1,\ldots,g_{n-1})$; so $H^0(G)=\mathbb{Z}, H^1(G)=G^{ab}=G/G^{comm}$.

One defines a geodesic simplex for three elements g_i of G by $\Delta(z)$ with (cf. (C.13))

$$\sigma((g_1, g_2, g_3)) = z = cr\{\infty, g_1\infty, g_1 g_2\infty, g_1 g_2 g_3\infty\}\,. \tag{D.2}$$

With ω a $\mathbb{C}$-valued $\mathrm{Sl}(2,\mathbb{C})$ invariant three-form on $\mathrm{Sl}(2,\mathbb{C})/\,\mathrm{SU}(2) = \mathbf{H}_3$ one finds a $\mathbb{C}/\mathbb{Z}$ valued Eilenberg-MacLane cochain $\mathcal{I}(\omega)$ with $\hat{C}_2 = \mathcal{I}(\omega)(g_1, g_2, g_3) = \int_{\Delta(z)}\omega$. This is evaluated [22] as $2\hat{C}_2 = c$ via the exterior square version (D.5) of the Heisenberg bundle

$$\begin{array}{ccccc}
 & & \mathbb{Q}\backslash\mathbb{C} & \longrightarrow & \mathbb{Q}\backslash\mathbb{C} \\
 & & \alpha\uparrow\downarrow & & \downarrow 1\wedge id \\
 & \overset{c}{\nearrow} & \widetilde{\mathbb{C}\wedge_{\mathbb{Z}}\mathbb{C}} & \longrightarrow & \mathbb{C}\wedge_{\mathbb{Z}}\mathbb{C} \\
 & & \rho\uparrow\downarrow & & \downarrow e \\
H^3(\mathrm{Sl}(2,\mathbb{C}))/(\mathbb{Q}/\mathbb{Z}) & \xrightarrow{\sigma} & \mathbf{P}^1\backslash\{0,1,\infty\} & \overset{z\wedge(1-z)}{\longrightarrow} & \mathbb{C}^*\wedge_{\mathbb{Z}}\mathbb{C}^*
\end{array} \tag{D.3}$$

(for the proper target of σ cf. (D.4)) The arrows in the lower row compose to zero.[82] This is the commutative diagram with exact rows[83] [22, 23]

$$\begin{array}{ccccc}
H^3(\mathrm{Sl}(2,\mathbb{C}),\mathbb{Z})/(\mathbb{Q}/\mathbb{Z}) & \overset{\sigma}{\hookrightarrow} & \mathbf{P_C} & \xrightarrow{\lambda} & \Lambda^2_{\mathbb{Z}}(\mathbb{C}^*) \\
\downarrow c & & \downarrow\rho & & || \\
\mathcal{C}/\mathbb{Q} & \overset{1\wedge id}{\hookrightarrow} & \Lambda^2_{\mathbb{Z}}(\mathbb{C}) & \xrightarrow{e} & \Lambda^2_{\mathbb{Z}}(\mathbb{C}^*)
\end{array} \tag{D.4}$$

with[84] $\mathbf{P}_{\mathbb{C}} = F(\mathbf{P}^1_{\mathbb{C}})/\Sigma_3^-$, i.e. free generators from $\mathbf{P}^1_{\mathbb{C}}$ modulo the equivalence relation given by the non-linear Σ_3 action with order two elements operating together with a minus sign (λ is then still welldefined).[85] Therefore $\rho\circ\sigma$ comes from an element in $\mathbb{Q}\backslash\mathbb{C}$, i.e. one defines $c = \alpha\circ\rho\circ\sigma$ (a natural continuous option for the splitting α is given only for the imaginary part); so c is essentially given by ρ, i.e. the Rogers 'function' (in the end the dilogarithm). One finds then that $2\hat{C}_2 = c$, more precisely[86] (6.14).

Representation via exterior squares. We are interested in the diagram analogous to (B.11) (again $e(z,w) = (e^{2\pi iz}, e^{2\pi iw})$; the tilde indicates a pullback of the e projection map along the indicated base map)

$$\begin{array}{ccc}
 & & \mathbb{Q}\backslash\mathbb{C} \\
 & & \downarrow 1\wedge id \\
\widetilde{\mathbb{C}\wedge_{\mathbb{Z}}\mathbb{C}} & \longrightarrow & \mathbb{C}\wedge_{\mathbb{Z}}\mathbb{C} \\
\downarrow p & & \downarrow e \\
\mathbf{P}^1\backslash\{0,1,\infty\} & \overset{z\wedge(1-z)}{\longrightarrow} & \mathbb{C}^*\wedge_{\mathbb{Z}}\mathbb{C}^*
\end{array} \tag{D.5}$$

[82] The target space of e has to be ('log'-)interpreted so that $(ab)\wedge c = a\wedge c + b\wedge c$, $\frac{1}{b}\wedge c = -(b\wedge c)$, $0 = \pm 1\wedge c$ hold; in particular $\lambda(z) = 0$ for $z\in\mu_{\mathbb{C}}$ (the complex roots of unity) as $z\wedge(1-z) = (1/n)(z^n\wedge(1-z))$.

[83] The upper row is well-defined as the part $\mathbb{Q}/\mathbb{Z}$ modded out comes from $H^3\left(\mu_{\mathbb{C}},\mathbb{Z}\right)$ embedded diagonally and this goes to zero under σ; note also that $\lambda(z) = 0$ for $z\in\mu_{\mathbb{C}}$ by footn. 82.

[84] In [22, 23] actually a group $\mathcal{P}_{\mathbb{C}} = \mathbf{P}_{\mathbb{C}}/\!\sim$ for a certain 5-term equivalence relation $\sim$ is considered.

[85] Note that the mapping $\lambda : \mathbb{C}\ni z\to z\wedge(1-z)\in\mathbb{C}^*\wedge_{\mathbb{Z}}\mathbb{C}^*$ transforms anti-invariantly (as $-\lambda(z) = z\wedge\beta z$ and $z\cdot\beta z\cdot\beta^2 z = -1$), i.e. $z\wedge(1-z)\in -\mathbf{1}$ (cf. section A.2).

[86] It suffices to evaluate $2\,\mathrm{Re}\,\hat{C}_2$ on the cohomology $H_3(\mathrm{Sl}(2,\mathbb{R})^\delta)$ of the real subgroup (actually the universal cover $\widetilde{\mathrm{PSl}}(2,\mathbb{R})^\delta$ is concerned) where one finds that in $H^3(\mathrm{Sl}(2,\mathbb{R}),\mathbb{R}/\mathbb{Z})$ it is congruent to $\frac{1}{4\pi^2}R(z)$ modulo $\frac{1}{24}$ $(=\frac{1}{4\pi^2}\cdot\frac{\pi^2}{6})$; $\mathrm{Im}\,\hat{C}_2$ is a *continuous* cochain and so uniquely determined (up to a factor) as $H^3_{\mathrm{cont}}(\mathrm{Sl}(2,\mathbb{C}),\mathbb{R})\cong\mathbb{R}$ [22].

Note that one has the expression which is *not* welldefined as a function (cf. (B.15))

$$\frac{1}{2\pi^2}\,\mathrm{Li}(z)=\frac{1}{2\pi i}\log z\frac{1}{2\pi i}\log(1-z)+\frac{-2}{(2\pi i)^2}R(z) \tag{D.6}$$

and can define a map $\rho:\mathbb{C}\ni z\longrightarrow\rho(z)\in\Lambda^2_{\mathbb{Z}}(\mathbb{C})$ which *is* indeed welldefined [20]

$$\rho(z)=\frac{1}{2\pi i}\log z\wedge\frac{1}{2\pi i}\log(1-z)+1\wedge\frac{-2}{(2\pi i)^2}R(z)\,. \tag{D.7}$$

This is a section of p in (D.5) just as (4.35) was a section of pr in (B.11). Note that if one wants to go back from a value in $\Lambda^2_{\mathbb{Z}}(\mathbb{C})$ to a complex number (to have a function instead of a section of a non-trivial projection), i.e. if one wants to define a splitting $\alpha:\mathbb{C}\wedge_{\mathbb{Z}}\mathbb{C}\rightarrow\mathbb{Q}\backslash\mathbb{C}$ to $\mathbb{Q}\backslash\mathbb{C}\overset{1\wedge id}{\hookrightarrow}\mathbb{C}\wedge_{\mathbb{Z}}\mathbb{C}$ one has *natural* option just for the imaginary part (this replaces (B.13); cf. also the alternative identification before (B.10)) [22, 23]

$$\mathrm{Im}\,\alpha(z\wedge w)=\mathrm{Re}\,z\,\mathrm{Im}\,w-\mathrm{Re}\,w\,\mathrm{Im}\,z\,. \tag{D.8}$$

The relative minus sign escapes the symmetry in (A.21); for by (A.26) ψ, and so $\mathcal{L}$, is not an imaginary part of ordinary (rather than wedge) products. One has[87] (cf. (B.14))

$$\mathrm{Im}\,\alpha\rho=\frac{1}{2\pi^2}\mathcal{L}\,. \tag{D.9}$$

References

[1] M. Atiyah, J.M. Maldacena and C. Vafa, *An M-theory flop as a large-N duality*, *J. Math. Phys.* **42** (2001) 3209 [hep-th/0011256].

[2] M. Atiyah and E. Witten, *M-theory dynamics on a manifold of G_2 holonomy*, *Adv. Theor. Math. Phys.* **6** (2003) 1 [hep-th/0107177].

[3] B.S. Acharya, *On realising $N=1$ super Yang-Mills in M-theory*, hep-th/0011089.

[4] A. Brandhuber, J. Gomis, S.S. Gubser and S. Gukov, *Gauge theory at large-N and new G_2 holonomy metrics*, *Nucl. Phys.* **B 611** (2001) 179 [hep-th/0106034].

[5] T.R. Taylor and C. Vafa, *RR flux on Calabi-Yau and partial supersymmetry breaking*, *Phys. Lett.* **B 474** (2000) 130 [hep-th/9912152].

[6] C. Beasley and E. Witten, *A note on fluxes and superpotentials in M-theory compactifications on manifolds of G_2 holonomy*, *J. High Energy Phys.* **07** (2002) 046 [hep-th/0203061].

[7] E. Witten, *Phases of $N=2$ theories in two dimensions*, *Nucl. Phys.* **B 403** (1993) 159 [hep-th/9301042].

[8] F. Cachazo, K.A. Intriligator and C. Vafa, *A large-N duality via a geometric transition*, *Nucl. Phys.* **B 603** (2001) 3 [hep-th/0103067].

[9] S. Gukov, *Solitons, superpotentials and calibrations*, *Nucl. Phys.* **B 574** (2000) 169 [hep-th/9911011];
S. Gukov, C. Vafa and E. Witten, *CFT's from Calabi-Yau four-folds*, *Nucl. Phys.* **B 584** (2000) 69 [hep-th/9906070], erratum *ibid.* **B 608** (2001) 477.

[87] Such a relation without taking the imaginary part would be inappropriate ($\mathcal{L}\neq\mathrm{Im}\,R$ as $\alpha\,\rho\neq\frac{1}{2\pi^2}R$).

[10] B.S. Acharya and B. Spence, *Flux, supersymmetry and M-theory on 7-manifolds*, hep-th/0007213.

[11] G. Curio and A. Krause, *Four-flux and warped heterotic M-theory compactifications*, *Nucl. Phys.* **B 602** (2001) 172 [hep-th/0012152].

[12] G. Curio and A. Krause, *G-fluxes and non-perturbative stabilisation of heterotic M-theory*, *Nucl. Phys.* **B 643** (2002) 131 [hep-th/0108220].

[13] M. Aganagic and C. Vafa, *Mirror symmetry, D-branes and counting holomorphic discs*, hep-th/0012041.

[14] M. Aganagic and C. Vafa, *Mirror symmetry and a G_2 flop*, hep-th/0105225.

[15] M. Aganagic, A. Klemm and C. Vafa, *Disk instantons, mirror symmetry and the duality web*, *Z. Naturforsch.* **A57** (2002) 1 [hep-th/0105045].

[16] A. Brandhuber, *G_2 holonomy spaces from invariant three-forms*, *Nucl. Phys.* **B 629** (2002) 393 [hep-th/0112113].

[17] E. Witten, *Five-brane effective action in M-theory*, *J. Geom. Phys.* **22** (1997) 103 [hep-th/9610234]

[18] R. Hain, *Classical polylogarithms*, alg-geom/9202022;
D. Ramakrishnan, *A regulator for curves via the Heisenberg group*, *Bull. Amer. Math. Soc.* **5**^{2} (1981) 191.

[19] R. Hain and R. MacPherson, *Higher logarithms*, *Ill. Journ. Math.* **34**^{2} (1990) 392.

[20] S. Bloch, *Applications of the dilogarithm function*, *Intl. Symp. on Algebraic Geometry*, Kyoto 1981, p. 103.

[21] J. Milnor, *Hyperbolic geometry: the first 150 years*, *Bull. Amer. Math. Soc.* **6** (1982) 9;
J.G. Ratcliffe, *Foundations of hyperbolic manifolds*, *Graduate Texts in Mathematics* **149** Springer 1994.

[22] J.L. Dupont, *The dilogarithm as a characteristic class for flat bundles*, *Journ. of Pure and Appl. Alg.* **44** (1987) 137.

[23] J.L. Dupont and C.-H. Sah, *Scissors congruences II*, *Journ. of Pure and Appl. Alg.* **25** (1982) 159.

[24] J.A. Harvey and G.W. Moore, *Superpotentials and membrane instantons*, hep-th/9907026.

[25] N. Seiberg and E. Witten, *Electric-magnetic duality, monopole condensation and confinement in $N = 2$ supersymmetric Yang-Mills theory*, *Nucl. Phys.* **B 426** (1994) 19 [hep-th/9407087], erratum *ibid.* **B 430** (1994) 485.

[26] M. Matone, *Instantons and recursion relations in $N = 2$ SUSY gauge theory*, *Phys. Lett.* **B 357** (1995) 342 [hep-th/9506102].

[27] G. Curio, A. Klemm, D. Lüst and S. Theisen, *On the vacuum structure of type-II string compactifications on Calabi-Yau spaces with H-fluxes*, *Nucl. Phys.* **B 609** (2001) 3 [hep-th/0012213].

[28] P. Hořava and E. Witten, *Heterotic and type-I string dynamics from eleven dimensions*, *Nucl. Phys.* **B 460** (1996) 506 [hep-th/9510209]; *Eleven-dimensional supergravity on a manifold with boundary*, *Nucl. Phys.* **B 475** (1996) 94 [hep-th/9603142];
E. Witten, *Strong coupling expansion of Calabi-Yau compactification*, *Nucl. Phys.* **B 471** (1996) 135 [hep-th/9602070].

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[29] B. Acharya and E. Witten, *Chiral fermions from manifolds of G_2 holonomy*, `hep-th/0109152`.

[30] G. Curio, A. Klemm, B. Körs and D. Lüst, *Fluxes in heterotic and type-II string compactifications*, *Nucl. Phys.* **B 620** (2002) 237 [`hep-th/0106155`].

[31] B.R. Greene, K. Schalm and G. Shiu, *Dynamical topology change in M-theory*, *J. Math. Phys.* **42** (2001) 3171 [`hep-th/0010207`].

[32] G. Tian and S.T. Yau, *Three dimensional algebraic manifolds with $c_1 = 0$ and $\chi = -6$*, in *Mathematical aspects of string theory*, Proc. San Diego 1986, p. 543.

[33] C. Vafa, *Superstrings and topological strings at large N*, *J. Math. Phys.* **42** (2001) 2798 [`hep-th/0008142`].

[34] E. Witten, *Branes and the dynamics of QCD*, *Nucl. Phys.* **B 507** (1997) 658 [`hep-th/9706109`].

[35] G. Curio, *Superpotential of the M-theory conifold and type-IIA string theory*, `hep-th/0212233`.

[36] G. Curio, B. Körs and D. Lüst, *Fluxes and branes in type-II vacua and M-theory geometry with G_2 and* Spin(7) *holonomy*, *Nucl. Phys.* **B 636** (2002) 197 [`hep-th/0111165`].

[37] D. Joyce, *On counting special lagrangian homology 3-spheres*, *Contemp. Math.* **314** (2002) 125 [`hep-th/9907013`].

[38] S. Kachru and J. McGreevy, *Supersymmetric three-cycles and (super)symmetry breaking*, *Phys. Rev.* **D 61** (2000) 026001 [`hep-th/9908135`].

[39] F. Denef, *Supergravity flows and D-brane stability*, *J. High Energy Phys.* **08** (2000) 050 [`hep-th/0005049`]; *(Dis)assembling special lagrangians*, `hep-th/0107152`.

[40] E. Witten, *2+1 dimensional gravity as an exactly soluble system*, *Nucl. Phys.* **B 311** (1988) 46.

[41] in preparation

[42] B. Richmond and G. Szekeres, *Some formulas related to dilogarithms, the zeta function and the Andrews-Gordon identities*, *J. Austral. Math. Soc.* **31** (1981) 362;
A.N. Kirillov and Yu.N. Reshetikhin, *Exact solutions of the integrable XXZ Heisenberg model with arbitrary spin: I, II*, *J. Phys.* **A 20** (1987) 1565, *J. Phys.* **A 20** (1987) 1587.

[43] J.L. Dupont and C.H. Sah, *Dilogarithm identities in conformal field theory and group homology*, *Commun. Math. Phys.* **161** (1994) 265 [`hep-th/9303111`].

[44] R. Caracciolo, F. Gliozzi and R. Tateo, *A topological invariant of RG flows in 2d integrable quantum field theories*, *Int. J. Mod. Phys.* **B 13** (1999) 2927 [`hep-th/9902094`].

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